

# LOCAL CONSTANTS FOR HEISENBERG REPRESENTATIONS

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**ABSTRACT.** We can attach a local constant to every finite dimensional continuous complex representation of a local Galois group of a non-archimedean local field  $F/\mathbb{Q}_p$  by Deligne and Langlands. Tate [19] gives an explicit formula for computing local constants for linear characters of  $F^\times$ , but there is no explicit formula of local constant for any arbitrary representation of a local Galois group. In this article we study Heisenberg representations of the absolute Galois group  $G_F$  of  $F$  and give invariant formulas of local constants for Heisenberg representations of dimension prime to  $p$ .

## 1. Introduction

Let  $F$  be a non-archimedean local field (i.e., finite extension of the  $p$ -adic field  $\mathbb{Q}_p$ , for some prime  $p$ ). Let  $\bar{F}$  be an algebraic closure of  $F$ , and  $G_F := \text{Gal}(\bar{F}/F)$  be the absolute Galois group of  $F$ . Let

$$\rho : G_F \rightarrow \text{Aut}_{\mathbb{C}}(V)$$

be a finite dimensional continuous complex representation of the Galois group  $G_F$ . For this  $\rho$ , we can associate a constant  $W(\rho)$  with absolute value 1 by Langlands (cf. [24]) and Deligne (cf. [28]). This constant is called the **local constant** (also known as local epsilon factor) of the representation  $\rho$ . Langlands also proves that these local constants are weakly extendible functions (cf. [19], p. 105, Theorem 1).

The existence of this local constant is proved by Tate for one-dimensional representation in [21] and the general proof of the existence of the local constants is proved by Langlands (see [24]). In 1972 Deligne also gave a proof using global methods in [28]. But in Deligne's terminology this local constant  $W(\rho)$  is  $\epsilon_D(\rho, \psi_F, dx, 1/2)$ , where  $dx$  is the Haar measure on  $F^+$  (locally compact abelian group) which is self-dual with respect to the **canonical** (i.e., coming through trace map from  $\psi_{\mathbb{Q}_p}(x) := e^{2\pi i x}$  for all  $x \in \mathbb{Q}_p$ , see [19], p. 92) additive character  $\psi_F$  of  $F$ . Tate in his article [20] denotes this Langlands convention of local constants as  $\epsilon_L(\rho, \psi)$ . According to Tate (cf. [20], p. 17), the Langlands factor  $\epsilon_L(\rho, \psi)$  is  $\epsilon_L(\rho, \psi) = \epsilon_D(\rho \omega_{\frac{1}{2}}, \psi, dx_\psi)$ , where  $\omega$  denotes the normalized absolute value of  $F$ , i.e.,  $\omega_{\frac{1}{2}}(x) = |x|_F^{\frac{1}{2}} = q_F^{-\frac{1}{2}\nu_F(x)}$  which we may consider as a character of  $F^\times$ , and where  $dx_\psi$  is the self-dual Haar measure corresponding to the additive character  $\psi$  and  $q_F$  is the cardinality of the residue field of  $F$ . According to Tate (cf. [19], p. 105) the relation among three conventions of the local constants is:

$$(1.1) \quad W(\rho) = \epsilon_L(\rho, \psi_F) = \epsilon_D(\rho \omega_{\frac{1}{2}}, \psi_F, dx_{\psi_F}).$$

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In Section 2, we discuss all the necessary notations and known results for this article. In Section 3 we study the arithmetic description of Heisenberg representations and their determinants (cf. Proposition 3.7) of the absolute Galois group  $G_F$  of a local field  $F/\mathbb{Q}_p$ . In particular, the Heisenberg representations of dimension prime to  $p$  are important for this article. In Subsection 3.2, we study the various properties (e.g., Artin conductors, Swan conductors, dimension) of Heisenberg representations of dimension prime to  $p$ .

In Section 4, firstly, we give an invariant formula of local constant for a Heisenberg representation  $\rho$  of the absolute Galois group  $G_F$  of a non-archimedean local field  $F/\mathbb{Q}_p$  (cf. Theorem 4.4). In Theorem 4.7, we give an invariant formula of local constant of a minimal conductor Heisenberg representation  $\rho$  of dimension prime to  $p$ . And when  $\rho$  is not minimal conductor but dimension is prime to  $p$ , we have Theorems 4.9, 4.11.

In Section 5, we also discuss Tate's root-of-unity criterion, and by applying this Tate's criterion we give some information about the dimension and Artin conductor of a Heisenberg representation (cf. Proposition 5.2).

## 2. Notations and Preliminaries

**2.1. Abelian Local Constants.** We have explicit formula of abelian local constants due to Tate (cf. [19], pp. 93-94). Let  $F$  be a non-archimedean local field. Let  $O_F$  be the ring of integers of the local field  $F$  and  $P_F = \pi_F O_F$  be a prime ideal in  $O_F$ , where  $\pi_F$  is a uniformizer, i.e., an element in  $P_F$  whose valuation is one, i.e.,  $v_F(\pi_F) = 1$ . The order of the residue field of  $F$  is  $q_F$ . Let  $U_F = O_F - P_F$  be the group of units in  $O_F$ . Let  $P_F^i = \{x \in F : v_F(x) \geq i\}$  and for  $i \geq 0$  define  $U_F^i = 1 + P_F^i$  (with proviso  $U_F^0 = U_F = O_F^\times$ ).

Let  $\chi$  be a character of  $F^\times$  with conductor  $a(\chi)$ , i.e., the smallest integer such that  $\chi$  is trivial on  $U_F^{a(\chi)}$ . Let  $\psi$  be an additive character of  $F$  with conductor  $n(\psi)$ , i.e.,  $\psi$  is trivial on  $P_F^{-n(\psi)}$ , nontrivial on  $P_F^{-n(\psi)-1}$ . Then the local constant of  $\chi$  is (cf. [19], p. 94):

$$(2.1) \quad W_F(\chi, \psi) = \chi(c) q_F^{-\frac{a(\chi)}{2}} \sum_{x \in \frac{U_F}{U_F^{a(\chi)}}} \chi^{-1}(x) \psi\left(\frac{x}{c}\right),$$

where  $c = \pi_F^{a(\chi)+n(\psi)}$ .

**Definition 2.1 (Different and Discriminant).** Let  $K/F$  be a finite separable extension of non-archimedean local field  $F$ . We define the **inverse different (or codifferent)**  $\mathcal{D}_{K/F}^{-1}$  of  $K$  over  $F$  to be  $\pi_K^{-d_{K/F}} O_K$ , where  $d_{K/F}$  is the largest integer (this is the exponent of the different  $\mathcal{D}_{K/F}$ ) such that

$$\mathrm{Tr}_{K/F}(\pi_K^{-d_{K/F}} O_K) \subseteq O_F,$$

where  $\mathrm{Tr}_{K/F}$  is the trace map from  $K$  to  $F$ . Then the **different** is defined by:

$$\mathcal{D}_{K/F} = \pi_K^{d_{K/F}} O_K$$

and the **discriminant**  $D_{K/F}$  is

$$D_{K/F} = N_{K/F}(\pi_K^{d_{K/F}}) O_F.$$

If  $K/F$  is tamely ramified, then

$$(2.2) \quad \nu_K(\mathcal{D}_{K/F}) = d_{K/F} = e_{K/F} - 1.$$

**2.2. Extendible functions.** Let  $G$  be any finite group. We denote  $R(G)$  the set of all pairs  $(H, \rho)$ , where  $H$  is a subgroup of  $G$  and  $\rho$  is a virtual representation of  $H$ . The group  $G$  acts on  $R(G)$  by means of

$$(H, \rho)^g = (H^g, \rho^g), \quad g \in G, \\ \rho^g(x) = \rho(gxg^{-1}), \quad x \in H^g := g^{-1}Hg$$

Furthermore we denote by  $\widehat{H}$  the set of all one dimensional representations of  $H$  and by  $R_1(G)$  the subset of  $R(G)$  of pairs  $(H, \chi)$  with  $\chi \in \widehat{H}$ . Here character  $\chi$  of  $H$  we mean always a **linear** character, i.e.,  $\chi : H \rightarrow \mathbb{C}^\times$ .

Now define a function  $\mathcal{F} : R_1(G) \rightarrow \mathcal{A}$ , where  $\mathcal{A}$  is a multiplicative abelian group with

$$(2.3) \quad \mathcal{F}(H, 1_H) = 1$$

and

$$(2.4) \quad \mathcal{F}(H^g, \chi^g) = \mathcal{F}(H, \chi)$$

for all  $(H, \chi)$ , where  $1_H$  denotes the trivial representation of  $H$ .

Here a function  $\mathcal{F}$  on  $R_1(G)$  means a function which satisfies the equation (2.3) and (2.4).

A function  $\mathcal{F}$  is said to be extendible if  $\mathcal{F}$  can be extended to an  $\mathcal{A}$ -valued function on  $R(G)$  satisfying:

$$(2.5) \quad \mathcal{F}(H, \rho_1 + \rho_2) = \mathcal{F}(H, \rho_1)\mathcal{F}(H, \rho_2)$$

for all  $(H, \rho_i) \in R(G), i = 1, 2$ , and if  $(H, \rho) \in R(G)$  with  $\dim \rho = 0$ , and  $\Delta$  is a subgroup of  $G$  containing  $H$ , then

$$(2.6) \quad \mathcal{F}(\Delta, \text{Ind}_H^\Delta \rho) = \mathcal{F}(H, \rho),$$

where  $\text{Ind}_H^\Delta \rho$  is the virtual representation of  $\Delta$  induced from  $\rho$ . In general, let  $\rho$  be a representation of  $H$  with  $\dim \rho \neq 0$ . We can define a zero dimensional representation of  $H$  by  $\rho$  and which is:  $\rho_0 := \rho - \dim \rho \cdot 1_H$ . So  $\dim \rho_0$  is zero, then now we use the equation (2.6) for  $\rho_0$  and we have,

$$(2.7) \quad \mathcal{F}(\Delta, \text{Ind}_H^\Delta \rho_0) = \mathcal{F}(H, \rho_0).$$

Now replace  $\rho_0$  by  $\rho - \dim \rho \cdot 1_H$  in the above equation (2.7) and we have

$$\begin{aligned} \mathcal{F}(\Delta, \text{Ind}_H^\Delta (\rho - \dim \rho \cdot 1_H)) &= \mathcal{F}(H, \rho - \dim \rho \cdot 1_H) \\ \implies \frac{\mathcal{F}(\Delta, \text{Ind}_H^\Delta \rho)}{\mathcal{F}(\Delta, \text{Ind}_H^\Delta 1_H)^{\dim \rho}} &= \frac{\mathcal{F}(H, \rho)}{\mathcal{F}(H, 1_H)^{\dim \rho}}. \end{aligned}$$

Therefore,

$$(2.8) \quad \begin{aligned} \mathcal{F}(\Delta, \text{Ind}_H^\Delta \rho) &= \left\{ \frac{\mathcal{F}(\Delta, \text{Ind}_H^\Delta 1_H)}{\mathcal{F}(H, 1_H)} \right\}^{\dim \rho} \cdot \mathcal{F}(H, \rho) \\ &= \lambda_H^\Delta(\mathcal{F})^{\dim \rho} \mathcal{F}(H, \rho), \end{aligned}$$

where

$$(2.9) \quad \lambda_H^\Delta(\mathcal{F}) := \frac{\mathcal{F}(\Delta, \text{Ind}_H^\Delta 1_H)}{\mathcal{F}(H, 1_H)}.$$

But by the definition of  $\mathcal{F}$ , we have  $\mathcal{F}(H, 1_H) = 1$ , so we can write

$$(2.10) \quad \lambda_H^\Delta(\mathcal{F}) = \mathcal{F}(\Delta, \text{Ind}_H^\Delta 1_H).$$

This  $\lambda_H^\Delta(\mathcal{F})$  is called **Langlands  $\lambda$ -function** (or simply  $\lambda$ -function) which is independent of  $\rho$ . A extendible function  $\mathcal{F}$  is called **strongly** extendible if it satisfies equation (2.5) and fulfills equation (2.6) for all  $(H, \rho) \in R(G)$ , and if the equation (2.6) is fulfilled only when  $\dim \rho = 0$ , then  $\mathcal{F}$  is called **weakly** extendible function. The extendible functions are **unique**, if they exist (cf. [19], p. 103).

**Example 2.2.** Langlands proves the local constants are weakly extendible functions (cf. [19], p. 105, Theorem 1). The Artin root numbers (also known as global constants) are strongly extendible functions (for more examples and details about extendible function, see [19] and [10]).

Now we take a tower of local Galois extensions  $K/L/F$ , and denote  $G = \text{Gal}(K/F)$ ,  $H = \text{Gal}(K/L)$ . Then the  $\lambda$ -function for the extension  $L/F$  is:

$$\lambda_{\text{Gal}(K/L)}^{\text{Gal}(K/F)}(W) := \lambda_{L/F}(\psi) = W(\text{Ind}_{L/F}(1_L), \psi),$$

where  $1_L$  is the trivial character of  $L^\times$  which corresponds to the trivial character of  $H$  by class field theory, and  $\psi$  is a nontrivial additive character of  $F$ . And when we take  $\psi = \psi_F$  as the canonical additive character, we simply write  $\lambda_{L/F}$  instead of  $\lambda_{L/F}(\psi_F)$ .

Since the Heisenberg representations of a finite local Galois are monomial (i.e., induced from linear character of a finite-index subgroup), we need to know the explicit formula for lambda functions for finite Galois extensions. For this article we need the following computations of lambda functions.

**Theorem 2.3** ([25], Theorem 3.5). *Let  $F$  be a non-archimedean local field and  $\text{Gal}(E/F)$  be a local Galois group of odd order. If  $L \supset K \supset F$  be any finite extension inside  $E$ , then  $\lambda_{L/K} = 1$ .*

**Theorem 2.4** ([25], Theorem 5.9). *Let  $K$  be a tamely ramified quadratic extension of  $F/\mathbb{Q}_p$  with  $q_F = p^s$ . Let  $\psi_F$  be the canonical additive character of  $F$ . Let  $c \in F^\times$  with  $-1 = \nu_F(c) + d_{F/\mathbb{Q}_p}$ , and  $c' = \frac{c}{\overline{\text{Tr}_{F/F_0}(\text{pc})}}$ , where  $F_0/\mathbb{Q}_p$  is the maximal unramified extension in  $F/\mathbb{Q}_p$ . Let  $\psi_{-1}$  be an additive character of  $F$  with conductor  $-1$ , of the form  $\psi_{-1} = c' \cdot \psi_F$ . Then*

$$\lambda_{K/F}(\psi_F) = \Delta_{K/F}(c') \cdot \lambda_{K/F}(\psi_{-1}),$$

where

$$\lambda_{K/F}(\psi_{-1}) = \begin{cases} (-1)^{s-1} & \text{if } p \equiv 1 \pmod{4} \\ (-1)^{s-1} i^s & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

If we take  $c = \pi_F^{-1-d_{F/\mathbb{Q}_p}}$ , where  $\pi_F$  is a norm for  $K/F$ , then

$$(2.11) \quad \Delta_{K/F}(c') = \begin{cases} 1 & \text{if } \overline{\text{Tr}_{F/F_0}(\text{pc})} \in k_{F_0}^\times = k_F^\times \text{ is a square,} \\ -1 & \text{if } \overline{\text{Tr}_{F/F_0}(\text{pc})} \in k_{F_0}^\times = k_F^\times \text{ is not a square.} \end{cases}$$

Here "overline" stands for modulo  $P_{F_0}$ .

**Theorem 2.5** ([25], Corollary 4.11(1)). *Let  $G = \text{Gal}(E/F)$  be a finite local Galois group of a non-archimedean local field  $F/\mathbb{Q}_p$  with  $p \neq 2$ . Let  $S \cong G/H$  be a nontrivial Sylow 2-subgroup of  $G$ , where  $H$  is a uniquely determined Hall subgroup of odd order. Suppose that we have a tower  $E/K/F$  of fields such that  $S \cong \text{Gal}(K/F)$ ,  $H = \text{Gal}(E/K)$  and  $G = \text{Gal}(E/F)$ . If  $S \subset G$  is cyclic, then*

(1)

$$\lambda_1^G = \lambda_{K/F}^{\pm 1} = \begin{cases} \lambda_{K/F} = W(\alpha) & \text{if } [E : K] \equiv 1 \pmod{4} \\ \lambda_{K/F}^{-1} = W(\alpha)^{-1} & \text{if } [E : K] \equiv -1 \pmod{4}, \end{cases}$$

(here  $\alpha = \Delta_{K/F}$  corresponds to the unique quadratic subextension in  $K/F$ ) if  $[K : F] = 2$ , hence  $\alpha = \Delta_{K/F}$ .

(2)

$$\lambda_1^G = \beta(-1)W(\alpha)^{\pm 1} = \beta(-1) \times \begin{cases} W(\alpha) & \text{if } [E : K] \equiv 1 \pmod{4} \\ W(\alpha)^{-1} & \text{if } [E : K] \equiv -1 \pmod{4} \end{cases}$$

if  $K/F$  is cyclic of order 4 with generating character  $\beta$  such that  $\beta^2 = \alpha = \Delta_{K/F}$ .

(3)

$$\lambda_1^G = \lambda_{K/F}^{\pm 1} = \begin{cases} \lambda_{K/F} = W(\alpha) & \text{if } [E : K] \equiv 1 \pmod{4} \\ \lambda_{K/F}^{-1} = W(\alpha)^{-1} & \text{if } [E : K] \equiv -1 \pmod{4} \end{cases}$$

if  $K/F$  is cyclic of order  $2^n \geq 8$ .

And if the 4th roots of unity are in the  $F$ , we have the same formulas as above but with 1 instead of  $\pm 1$ . Moreover, when  $p \neq 2$ , a precise formula for  $W(\alpha)$  will be obtained in Theorem 2.4.

**2.3. Classical Gauss sums.** Let  $k_q$  be a finite field of order  $q$ . Let  $\chi, \psi$  be a multiplicative and an additive character respectively of  $k_q$ . Then the Gauss sum  $G(\chi, \psi)$  is defined by

$$(2.12) \quad G(\chi, \psi) = \sum_{x \in k_q^\times} \chi(x)\psi(x).$$

For this article we need the following theorem. In general, we cannot give explicit formula of  $G(\chi, \psi)$  for arbitrary character  $\chi$ . But if  $q = p^r$  ( $r \geq 2$ ), where  $p$  is an odd prime, then by R. Odoni (cf. [1], p. 33, Theorem 1.6.2) we can show that  $G(\chi, \psi)/\sqrt{q}$  is a certain root of unity. If  $q$  is an odd prime and order of  $\chi$  is  $\geq 3$ , then  $G(\chi, \psi)/\sqrt{q}$  is **not** a root of unity.

**Theorem 2.6** (Chowla, [1], p. 31, Theorem 1.6.1). *Let  $q$  be an odd prime, and let  $\chi$  be a character of  $k_q^\times$  of order  $> 2$ . Let  $\psi(x) = e^{\frac{2\pi i x}{q}}$  for  $x \in k_q$ . Then the Gauss sum  $G(\chi, \psi)$  does not equal to  $\sqrt{q}$  times a root of unity,*

**2.4. Heisenberg representation.** Let  $\rho$  be an irreducible representation of a (pro-)finite group  $G$ . Then  $\rho$  is called a **Heisenberg representation** if it represents commutators by scalar matrices. Therefore higher commutators are represented by 1. We can see that the linear characters of  $G$  are Heisenberg representations as the degenerate special case. To classify Heisenberg representations we need to mention two invariants of an irreducible representation  $\rho \in \text{Irr}(G)$ :

- (1) Let  $Z_\rho$  be the **scalar** group of  $\rho$ , i.e.,  $Z_\rho \subseteq G$  and  $\rho(z) = \text{scalar matrix}$  for every  $z \in Z_\rho$ . If  $V/\mathbb{C}$  is a representation space of  $\rho$  we get  $Z_\rho$  as the kernel of the composite map

$$(2.13) \quad G \xrightarrow{\rho} GL_{\mathbb{C}}(V) \xrightarrow{\pi} PGL_{\mathbb{C}}(V) = GL_{\mathbb{C}}(V)/\mathbb{C}^\times E,$$

where  $E$  is the unit matrix and denote  $\bar{\rho} := \pi \circ \rho$ . Therefore  $Z_\rho$  is a normal subgroup of  $G$ .

- (2) Let  $\chi_\rho$  be the character of  $Z_\rho$  which is given as  $\rho(g) = \chi_\rho(g) \cdot E$  for all  $g \in Z_\rho$ . Apparently  $\chi_\rho$  is a  $G$ -invariant character of  $Z_\rho$  which we call the central character of  $\rho$ .

Let  $A$  be a profinite abelian group. Then we know that (cf. [8], p. 124, Theorem 1 and Theorem 2) the set of isomorphism classes  $\text{PI}(A)$  of projective irreducible representations (for projective representation, see [3], §51) of  $A$  is in bijective correspondence with the set of continuous alternating characters  $\text{Alt}(A)$ . If  $\rho \in \text{PI}(A)$  corresponds to  $X \in \text{Alt}(A)$  then

$$\text{Ker}(\rho) = \text{Rad}(X) \quad \text{and} \quad [A : \text{Rad}(X)] = \dim(\rho)^2,$$

where  $\text{Rad}(X) := \{a \in A \mid X(a, b) = 1, \text{ for all } b \in A\}$ , the **radical of  $X$** .

Let  $A := G/[G, G]$ , so  $A$  is abelian. We also know from the composite map (2.13)  $\bar{\rho}$  is a projective irreducible representation of  $G$  and  $Z_\rho$  is the kernel of  $\bar{\rho}$ . Therefore **modulo commutator group**  $[G, G]$ , we can consider that  $\bar{\rho}$  is in  $\text{PI}(A)$  which corresponds an alternating character  $X$  of  $A$  with kernel of  $\bar{\rho}$  is  $Z_\rho/[G, G] = \text{Rad}(X)$ . We also know that

$$[A : \text{Rad}(X)] = [G/[G, G] : Z_\rho/[G, G]] = [G : Z_\rho].$$

Then we observe that

$$\dim(\bar{\rho}) = \dim(\rho) = \sqrt{[G : Z_\rho]}.$$

Let  $H$  be a subgroup of  $A$ , then we define the orthogonal complement of  $H$  in  $A$  with respect to  $X$

$$H^\perp := \{a \in A : X(a, H) \equiv 1\}.$$

An **isotropic** subgroup  $H \subset A$  is a subgroup such that  $H \subseteq H^\perp$  (cf. [4], p. 270, Lemma 1(v)). And when isotropic subgroup  $H$  is maximal, we call  $H$  is a **maximal isotropic** for  $X$ . Thus when  $H$  is maximal isotropic we have  $H = H^\perp$ .

We also can show that the Heisenberg representations  $\rho$  are fully characterized by the corresponding pair  $(Z_\rho, \chi_\rho)$ .

**Proposition 2.7** ([6], **Proposition 4.2**). *The map  $\rho \mapsto (Z_\rho, \chi_\rho)$  is a bijection between equivalence classes of Heisenberg representations of  $G$  and the pairs  $(Z_\rho, \chi_\rho)$  such that*

- (a)  $Z_\rho \subseteq G$  is a coabelian normal subgroup,
- (b)  $\chi_\rho$  is a  $G$ -invariant character of  $Z_\rho$ ,
- (c)  $X(\hat{g}_1, \hat{g}_2) := \chi_\rho(g_1 g_2 g_1^{-1} g_2^{-1})$  is a nondegenerate **alternating character** on  $G/Z_\rho$  where  $\hat{g}_1, \hat{g}_2 \in G/Z_\rho$  and their corresponding lifts  $g_1, g_2 \in G$ .

For pairs  $(Z_\rho, \chi_\rho)$  with the properties (a)–(c), the corresponding Heisenberg representation  $\rho$  is determined by the identity (cf. [27], p. 30):

$$(2.14) \quad \sqrt{[G : Z_\rho]} \cdot \rho = \text{Ind}_{Z_\rho}^G \chi_\rho.$$

Let  $C^1 G = G$ ,  $C^{i+1} G = [C^i G, G]$  denote the descending central series of  $G$ . Now assume that every projective representation of  $A$  lifts to an ordinary representation of  $G$ . Then by I. Schur's results (cf. [3], p. 361, Theorem 53.7) we have (cf. [8], p. 124, Theorem 2):

(1) Let  $A \wedge_{\mathbb{Z}} A$  denote the alternating square of the  $\mathbb{Z}$ -module  $A$ . The commutator map

$$(2.15) \quad A \wedge_{\mathbb{Z}} A \cong C^2 G / C^3 G, \quad a \wedge b \mapsto [\hat{a}, \hat{b}]$$

is an isomorphism.

(2) The map  $\rho \rightarrow X_\rho \in \text{Alt}(A)$  from Heisenberg representations to alternating characters on  $A$  is surjective.

*Remark 2.8.* Let  $\chi_\rho$  be a character of  $Z_\rho$ . All extensions  $\chi_H \supset \chi_\rho$  are conjugate with respect to  $G/H$ . This can be easily seen, since we know  $\chi_H \supset \chi_\rho$  and  $\chi_H^g(h) = \chi_H(ghg^{-1})$ . If we take  $z \in Z_\rho$ , then we obtain

$$\begin{aligned} \chi_H^g(z) &= \chi_H(gzg^{-1}) = \chi_\rho(gzg^{-1}) = \chi_\rho(gzg^{-1}z^{-1}z) \\ &= \chi_\rho([g, z]z) = X(g, z) \cdot \chi_\rho(z) = \chi_\rho(z), \end{aligned}$$

since  $Z_\rho$  is a normal subgroup of  $G$  and the radical of  $X$  (i.e.,  $X(g, z) = \chi_\rho([g, z]) = 1$  for all  $z \in Z_\rho$  and  $g \in G$ ). Therefore,  $\chi_H^g$  are extensions of  $\chi_\rho$  for all  $g \in G/H$ . It can also be seen that the conjugates  $\chi_H^g$  are all different, because  $\chi_H^{g_1} = \chi_H^{g_2}$  is the same as  $\chi_H^{g_1 g_2^{-1}} = \chi_H$ . So it is enough to see that  $\chi_H^{g^{-1}} \neq 1$  if  $g \neq 1 \in G/H$ . But

$$\chi_H^{g^{-1}}(h) = \chi_\rho(ghg^{-1}h^{-1}) = X(g, h),$$

and therefore  $\chi_H^{g^{-1}} \equiv 1$  on  $H$  implies  $g \in H^\perp = H$ , where “ $\perp$ ” denotes the orthogonal complement with respect to  $X$ . Then for a given one extension  $\chi_H$  of  $\chi_\rho$  all other extensions are of the form  $\chi_H^g$  for  $g \in G/H$ .

*Remark 2.9.* Let  $\rho = (Z, \chi_\rho)$  be a Heisenberg representation of  $G$ . Then from the definition of Heisenberg representation we have

$$[[G, G], G] \subseteq \text{Ker}(\rho).$$

Now let  $\overline{G} := G/\text{Ker}(\rho)$ . Then we obtain

$$[\overline{G}, \overline{G}] = [G/\text{Ker}(\rho), G/\text{Ker}(\rho)] = [G, G] \cdot \text{Ker}(\rho)/\text{Ker}(\rho) = [G, G]/[G, G] \cap \text{Ker}(\rho).$$

Since  $[[G, G], G] \subseteq \text{Ker}(\rho)$ , then  $[x, g] \in \text{Ker}(\rho)$  for all  $x \in [G, G]$  and  $g \in G$ . Hence we obtain

$$[[\overline{G}, \overline{G}], \overline{G}] = [[G, G]/[G, G] \cap \text{Ker}(\rho), G/\text{Ker}(\rho)] \subseteq \text{Ker}(\rho),$$

This shows that  $\overline{G}$  is a two-step nilpotent group.

### 3. Arithmetic description of Heisenberg representations

In Section 2.5, we see the notion of Heisenberg representations of a (pro-)finite group. These Heisenberg representations have arithmetic structure due to E.-W. Zink (cf. [5], [7], [8]). For this article we need to describe the arithmetic structure of Heisenberg representations.

Let  $F/\mathbb{Q}_p$  be a local field, and  $\bar{F}$  be an algebraic closure of  $F$ . Denote  $G_F = \text{Gal}(\bar{F}/F)$  the absolute Galois group for  $\bar{F}/F$ . We know that (cf. [11], p. 197) each representation  $\rho : G_F \rightarrow GL(n, \mathbb{C})$  corresponds to a projective representation  $\bar{\rho} : G_F \rightarrow GL(n, \mathbb{C}) \rightarrow PGL(n, \mathbb{C})$ . On the other hand, each projective representation  $\bar{\rho} : G_F \rightarrow PGL(n, \mathbb{C})$  can be lifted to a representation  $\rho : G_F \rightarrow GL(n, \mathbb{C})$ . Let  $A_F = G_F^{ab}$  be the factor commutator group of  $G_F$ . Define

$$FF^\times := \varprojlim (F^\times/N \wedge F^\times/N)$$

where  $N$  runs over all open subgroups of finite index in  $F^\times$ . Denote by  $\text{Alt}(F^\times)$  as the set of all alternating characters  $X : F^\times \times F^\times \rightarrow \mathbb{C}^\times$  such that  $[F^\times : \text{Rad}(X)] < \infty$ . Then the local reciprocity map gives an isomorphism between  $A_F$  and the profinite completion of  $F^\times$ , and induces a natural bijection

$$(3.1) \quad \text{PI}(A_F) \xrightarrow{\sim} \text{Alt}(F^\times),$$

where  $\text{PI}(A_F)$  is the set of isomorphism classes of projective irreducible representations of  $A_F$ . By using class field theory from the commutator map (2.15) (cf. p. 125 of [8]) we obtain

$$(3.2) \quad c : FF^\times \cong [G_F, G_F]/[[G_F, G_F], G_F].$$

Let  $K/F$  be an abelian extension corresponding to the norm subgroup  $N \subset F^\times$  and if  $W_{K/F}$  denotes the relative Weil group, the commutator map for  $W_{K/F}$  induces an isomorphism (cf. p. 128 of [8]):

$$(3.3) \quad c : F^\times/N \wedge F^\times/N \rightarrow K_F^\times/I_F K^\times,$$

where

$$K_F^\times := \{x \in K^\times \mid N_{K/F}(x) = 1\}, \text{ i.e., the norm-1-subgroup of } K^\times,$$

$$I_F K^\times := \{x^{1-\sigma} \mid x \in K^\times, \sigma \in \text{Gal}(K/F)\} < K_F^\times, \text{ the augmentation with respect to } K/F.$$

Taking the projective limit over all abelian extensions  $K/F$  the isomorphisms (3.3) induce:

$$(3.4) \quad c : FF^\times \cong \varprojlim K_F^\times/I_F K^\times,$$

where the limit on the right side refers to norm maps. This gives an arithmetic description of Heisenberg representations of the group  $G_F$ .

**Theorem 3.1** (Zink, [5], p. 301, Corollary 1.2). *The set of Heisenberg representations  $\rho$  of  $G_F$  is in bijective correspondence with the set of all pairs  $(X_\rho, \chi_\rho)$  such that:*

- (1)  $X_\rho$  is a character of  $FF^\times$ ,
- (2)  $\chi_\rho$  is a character of  $K^\times/I_F K^\times$ , where the abelian extension  $K/F$  corresponds to the radical  $N \subset F^\times$  of  $X_\rho$ , and
- (3) via (3.3) the alternating character  $X_\rho$  corresponds to the restriction of  $\chi_\rho$  to  $K_F^\times$ .



Given a pair  $(X, \chi)$ , we can construct the Heisenberg representation  $\rho$  by induction from  $G_K := \text{Gal}(\overline{F}/K)$  to  $G_F$ :

$$(3.5) \quad \sqrt{[F^\times : N]} \cdot \rho = \text{Ind}_{K/F}(\chi),$$

where  $N$  and  $K$  are as in (2) of the above Theorem 3.1 and where the induction of  $\chi$  (to be considered as a character of  $G_K$  by class field theory) produces a multiple of  $\rho$ . From  $[F^\times : N] = [K : F]$  we obtain the **dimension formula**:

$$(3.6) \quad \dim(\rho) = \sqrt{[F^\times : N]},$$

where  $N$  is the radical of  $X$ .

Let  $K/E$  be an extension of  $E$ , and  $\chi_K : K^\times \rightarrow \mathbb{C}^\times$  be a character of  $K^\times$ . In the following lemma, we give the conditions of the existence of characters  $\chi_E \in \widehat{E}^\times$  such that  $\chi_E \circ N_{K/E} = \chi_K$ , and the solutions set of this  $\chi_E$ .

**Lemma 3.2.** *Let  $K/E$  be a finite extension of a field  $E$ , and  $\chi_K : K^\times \rightarrow \mathbb{C}^\times$ .*

- (i) *The existence of characters  $\chi_E : E^\times \rightarrow \mathbb{C}^\times$  such that  $\chi_E \circ N_{K/E} = \chi_K$  is equivalent to  $K_E^\times \subset \text{Ker}(\chi_K)$ .*
- (ii) *In case (i) is fulfilled, we have a well defined character*

$$(3.7) \quad \chi_{K/E} := \chi_K \circ N_{K/E}^{-1} : \mathcal{N}_{K/E} \rightarrow \mathbb{C}^\times,$$

*on the subgroup of norms  $\mathcal{N}_{K/E} := N_{K/E}(K^\times) \subset E^\times$ , and the solutions  $\chi_E$  such that  $\chi_E \circ N_{K/E} = \chi_K$  are precisely the extensions of  $\chi_{K/E}$  from  $\mathcal{N}_{K/E}$  to a character of  $E^\times$ .*

*Proof.* (i) Suppose that an equation  $\chi_K = \chi_E \circ N_{K/E}$  holds. Let  $x \in K_E^\times$ , hence  $N_{K/E}(x) = 1$ . Then

$$\chi_K(x) = \chi_E \circ N_{K/E}(x) = \chi_E(1) = 1.$$

So  $x \in \text{Ker}(\chi_K)$ , and hence  $K_E^\times \subset \text{Ker}(\chi_K)$ .

Conversely assume that  $K_E^\times \subset \text{Ker}(\chi_K)$ . Then  $\chi_K$  is actually a character of  $K^\times/K_E^\times$ . Again we have  $K^\times/K_E^\times \cong \mathcal{N}_{K/E} \subset E^\times$ , hence  $\widehat{K^\times/K_E^\times} \cong \widehat{\mathcal{N}_{K/E}}$ . Now suppose that  $\chi_K$  corresponds to the character  $\chi_{K/E}$  of  $\mathcal{N}_{K/E}$ . Hence we can write  $\chi_K \circ N_{K/E}^{-1} = \chi_{K/E}$ . Thus the character  $\chi_{K/E} : \mathcal{N}_{K/E} \rightarrow \mathbb{C}^\times$  is well defined. Since  $E^\times$  is an abelian group and  $\mathcal{N}_{K/E} \subset E^\times$  is a subgroup of finite index (by class field theory)  $[K : E]$ , we can extend  $\chi_{K/E}$  to  $E^\times$ , and  $\chi_K$  is of the form  $\chi_K = \chi_E \circ N_{K/E}$  with  $\chi_E|_{\mathcal{N}_{K/E}} = \chi_{K/E}$ .

(ii) If condition (i) is satisfied, then this part is obvious. If  $\chi_E$  is a solution of  $\chi_K = \chi_E \circ N_{K/E}$ , with  $\chi_{K/E} := \chi_K \circ N_{K/E}^{-1} : \mathcal{N}_{K/E} \rightarrow \mathbb{C}^\times$ , then certainly  $\chi_E$  is an extension of the character  $\chi_{K/E}$ .

Conversely, if  $\chi_E$  extends  $\chi_{K/E}$ , then it is a solution of  $\chi_K = \chi_E \circ N_{K/E}$  with  $\chi_K \circ N_{K/E}^{-1} = \chi_{K/E} : \mathcal{N}_{K/E} \rightarrow \mathbb{C}^\times$ .  $\square$

*Remark 3.3.* Now take Heisenberg representation  $\rho = \rho(X, \chi_K)$  of  $G_F$ . Let  $E/F$  be any extension corresponding to a maximal isotropic for  $X$ . In this Heisenberg setting, from Theorem 3.1(2), we know  $\chi_K$  is a character of  $K^\times/I_F K^\times$ , and from the first commutative

diagram on p. 302 of [5] we have  $N_{K/E} : K_F^\times / I_F K^\times \rightarrow E_F^\times / I_F \mathcal{N}_{K/E}$ . Thus in the Heisenberg setting, we have more information than Lemma 3.2(i), that  $\chi_K$  is a character of

$$(3.8) \quad K^\times / K_E^\times I_F K^\times \xrightarrow{N_{K/E}} \mathcal{N}_{K/E} / I_F \mathcal{N}_{K/E} \subset E^\times / I_F \mathcal{N}_{K/E},$$

and therefore  $\chi_{K/F}$  is actually a character of  $\mathcal{N}_{K/E} / I_F \mathcal{N}_{K/E}$ , or in other words, it is a  $\text{Gal}(E/F)$ -invariant character of the  $\text{Gal}(E/F)$ -module  $\mathcal{N}_{K/E} \subset E^\times$ . And if  $\chi_E$  is one of the solution of Lemma 3.2(ii), then the complete solutions is the set  $\{\chi_E^\sigma \mid \sigma \in \text{Gal}(E/F)\}$ .

**We know that  $W(\chi_E, \psi \circ \text{Tr}_{K/E})$  has the same value for all solutions  $\chi_E$  of  $\chi_E \circ N_{K/E} = \chi_K$ , which means for all  $\chi_E$  which extend the character  $\chi_{K/E}$ .**

Moreover, from the above Lemma 3.2, we also can see that  $\chi_E|_{\mathcal{N}_{K/E}} = \chi_K \circ N_{K/E}^{-1}$ .

Let  $\rho = \rho(X, \chi_K)$  be a Heisenberg representation of  $G_F$ . Let  $E/F$  be any extension corresponding to a maximal isotropic for  $X$ . Then by using the above Lemma 3.2, we have the following lemma.

**Lemma 3.4.** *Let  $\rho = \rho(Z, \chi_\rho) = \rho(\text{Gal}(L/K), \chi_K)$  be a Heisenberg representation of a finite local Galois group  $G = \text{Gal}(L/F)$ , where  $F$  is a non-archimedean local field. Let  $H = \text{Gal}(L/E)$  be a maximal isotropic for  $\rho$ . Then we obtain*

$$(3.9) \quad \rho = \text{Ind}_{E/F}(\chi_E^\sigma) \quad \text{for all } \sigma \in \text{Gal}(E/F),$$

where  $\chi_E : E^\times / I_F \mathcal{N}_{K/E} \rightarrow \mathbb{C}^\times$  with  $\chi_K = \chi_E \circ N_{K/E}$ .

Moreover, for a fixed base field  $E$  of a maximal isotropic for  $\rho$ , this construction of  $\rho$  is independent of the choice of this character  $\chi_E$ .

*Proof.* From the group theoretical construction of Heisenberg representation (cf. see Section 2.6), we can write

$$(3.10) \quad \rho = \text{Ind}_H^G(\chi_H^g), \quad \text{for all } g \in G/H,$$

where  $\chi_H : H \rightarrow \mathbb{C}^\times$  is an extension of  $\chi_\rho$ . From Remark 2.8 we know that all extensions of character  $\chi_\rho$  are conjugate with respect to  $G/H$ , and they are different. If we fix  $H$ , then  $\rho$  is independent of the choice of character  $\chi_H$ . For every extension of  $\chi_\rho$  we will have same  $\rho$ . The assertion of the lemma is the arithmetic expression of this group theoretical facts, and which we will prove in the following.

By the given conditions,  $L/F$  is a finite Galois extension of the local field  $F$  and  $G = \text{Gal}(L/F)$ , and  $H = \text{Gal}(L/E)$ ,  $Z = \text{Gal}(L/K)$  and  $\{1\} = \text{Gal}(L/L)$ . Then by class field theory, equation (3.3), and the condition  $X := \chi_K \circ [-, -]$ ,  $\chi_\rho$  identifies with a character

$$\chi_K : K^\times / I_F K^\times \rightarrow \mathbb{C}^\times.$$

Moreover, for the Heisenberg representations we also have the following commutative diagram

$$(3.11) \quad \begin{array}{ccc} K_E^\times / I_E K^\times & \xrightarrow{\text{inclusion}} & K_F^\times / I_F K^\times \\ \uparrow c & & \uparrow c \\ E^\times / \mathcal{N}_{K/E} \wedge E^\times / \mathcal{N}_{K/E} & \xrightarrow{N_{E/F} \wedge N_{E/F}} & F^\times / \mathcal{N}_{K/F} \wedge F^\times / \mathcal{N}_{K/F} \end{array}$$

where  $N_{E/F} \wedge N_{E/F}(a \wedge b) = N_{E/F}(a) \wedge N_{E/F}(b)$  for all  $a, b \in E^\times$ , and the vertical isomorphisms in upward direction are given as the commutator maps (cf. equation (3.3)) in the Weil groups  $W_{K/E}/I_E K^\times$  and  $W_{K/F}/I_F K^\times$  respectively. Under the right vertical  $\chi_K$  corresponds (cf. Theorem 3.1(3)) to the alternating character  $X$  which is trivial on  $N_{E/F} \wedge N_{E/F}$ , because  $H$  corresponding to  $E^\times$  is isotropic. The commutative diagram now shows that  $\chi_K$  must be trivial on the image of the upper horizontal, i.e.,  $\chi_K$  is trivial on the subgroups  $K_E^\times$  for all maximal isotropic  $E$ . Hence  $\chi_K$  is actually a character of  $K^\times/K_E^\times$ .

Then from Lemma 3.2 we can say that there exists a character  $\chi_E : E^\times/I_F \mathcal{N}_{K/E} \rightarrow \mathbb{C}^\times$  such that  $\chi_K = \chi_E \circ N_{K/E}$ . And this  $\chi_E$  is determined by the character  $\chi_H$ . For  $\sigma \in G/H = \text{Gal}(E/F)$  we have  $\chi_E^\sigma \circ N_{K/E} = \chi_E \circ N_{K/E} = \chi_K$  because  $\chi_E^{\sigma^{-1}} \circ N_{K/E} \equiv 1$ , because  $\chi_E$  is trivial on  $I_F \mathcal{N}_{K/E}$ .

Therefore instead of  $\rho = \text{Ind}_H^G(\chi_H^g)$  for all  $g \in G/H$ , we obtain

$$\rho = \text{Ind}_{E/F}(\chi_E^\sigma), \text{ for all } \sigma \in \text{Gal}(E/F),$$

independently of the choice of  $\chi_E$ .

□

*Remark 3.5.* Moreover we have the exact sequence

$$(3.12) \quad K^\times/I_F K^\times \xrightarrow{N_{K/E}} E^\times/I_F \mathcal{N}_{K/E} \xrightarrow{N_{E/F}} F^\times/\mathcal{N}_{K/F},$$

which is only exact in the middle term. For the dual groups this gives

$$(3.13) \quad \widehat{K^\times/I_F K^\times} \xleftarrow{N_{K/E}^*} \widehat{E^\times/I_F \mathcal{N}_{K/E}} \xleftarrow{N_{E/F}^*} \widehat{F^\times/\mathcal{N}_{K/F}}.$$

But  $N_{K/E}^*(\chi_E^{\sigma^{-1}}) = \chi_E^{\sigma^{-1}} \circ N_{K/E} \equiv 1$ , and therefore the exactness of sequence (3.13) yields

$$(3.14) \quad \chi_E^{\sigma^{-1}} = \chi_F \circ N_{E/F}, \quad \text{for some } \chi_F \in \widehat{F^\times/\mathcal{N}_{K/F}},$$

For our (arithmetic) determinant computation of Heisenberg representation  $\rho$  of  $G_F$ , we need the following lemma regarding transfer map.

**Lemma 3.6.** *Let  $\rho = \rho(Z, \chi_\rho)$  be a Heisenberg representation of a group  $G$  and assume that  $H/Z \subset G/Z$  is a maximal isotropic for  $\rho$ . Then transfer map  $T_{(G/Z)/(H/Z)} \equiv 1$  is the trivial map.*

*Proof.* In general, if  $H$  is a central subgroup<sup>1</sup> of finite index  $n = [G : H]$  of a group  $G$ , then by Theorem 5.6 on p. 154 of [13] we have  $T_{G/H}(g) = g^n$ . If  $G$  is abelian, then center  $Z(G) = G$ . Hence every subgroup of  $G$  is central subgroup. Now if we take  $G$  as an abelian group and  $H$  is a subgroup of finite index, then we can write  $T_{G/H}(g) = g^{[G:H]}$ .

Now we come to the Heisenberg setting. We know that  $G/Z$  is abelian, hence  $H/Z \subset G/Z$  is a central subgroup. Then we have  $T_{(G/Z)/(H/Z)}(g) = g^{[G/Z:H/Z]} = g^d$ , where  $d$  is the dimension of  $\rho$ . For the Heisenberg setting, we also know (cf. Lemma 3.3 on p. 8 of [26]) that  $G^d \subseteq Z$ , hence  $g^d \in Z$ . This implies

$$T_{(G/Z)/(H/Z)}(g) = g^d = 1, \quad \text{the identity in } H/Z,$$

---

<sup>1</sup>A subgroup of a group which lies inside the center of the group, i.e., a subgroup  $H$  of  $G$  is central if  $H \subseteq Z(G)$ .

for all  $g \in G$ , hence  $T_{(G/Z)/(H/Z)} \equiv 1$  is a trivial map.  $\square$

By using the above Lemma 3.2 and Lemma 3.6, in the following, we give the arithmetic description of the determinant of Heisenberg representations.

**Proposition 3.7.** *Let  $\rho = \rho(Z, \chi_\rho) = \rho(G_K, \chi_K)$  be a Heisenberg representation of the absolute Galois group  $G_F$ . Let  $E$  be a base field of a maximal isotropic for  $\rho$ . Then  $F^\times \subseteq \mathcal{N}_{K/E}$ , and*

$$(3.15) \quad \det(\rho)(x) = \Delta_{E/F}(x) \cdot \chi_K \circ N_{K/E}^{-1}(x) \quad \text{for all } x \in F^\times,$$

where, for all  $x \in F^\times$ ,

$$(3.16) \quad \Delta_{E/F}(x) = \begin{cases} 1 & \text{when } \text{rk}_2(\text{Gal}(E/F)) \neq 1 \\ \omega_{E'/F}(x) & \text{when } \text{rk}_2(\text{Gal}(E/F)) = 1, \end{cases}$$

where  $E'/F$  is a uniquely determined quadratic subextension in  $E/F$ , and  $\omega_{E'/F}$  is the character of  $F^\times$  which corresponds to  $E'/F$  by class field theory.

*Proof.* From the given condition, we can write  $G/Z = \text{Gal}(K/F) \supset H/Z = \text{Gal}(K/E)$ . Here both  $G/Z$  and  $H/Z$  are abelian, then from class field theory we have the following commutative diagram

$$(3.17) \quad \begin{array}{ccc} F^\times / \mathcal{N}_{K/F} & \xrightarrow{\text{inclusion}} & E^\times / \mathcal{N}_{K/E} \\ \downarrow \theta_{K/F} & & \downarrow \theta_{K/E} \\ \text{Gal}(K/F) & \xrightarrow{T_{(G/Z)/(H/Z)}} & \text{Gal}(K/E) \end{array}$$

Here  $\theta_{K/F}, \theta_{K/E}$  are the isomorphism (Artin reciprocity) maps and  $T_{(G/Z)/(H/Z)}$  is transfer map. From Lemma 3.6, we have  $T_{(G/Z)/(H/Z)} \equiv 1$ . Therefore from the above diagram (3.17) we can say  $F^\times \subseteq \mathcal{N}_{K/E}$ , i.e., all elements <sup>2</sup> from the base field  $F$  are norms with respect to the extension  $K/E$ .

Now identify  $\chi_\rho = \chi_K : K^\times / I_F K^\times \rightarrow \mathbb{C}^\times$ . Then the map

$$x \in F^\times \mapsto \chi_K \circ N_{K/E}^{-1}(x)$$

is well-defined character of  $F^\times$ .

Now by Gallagher's Theorem (cf. [9], Theorem 30.1.6) (arithmetic side) we can write for all  $x \in F^\times$ ,

$$(3.18) \quad \det(\rho)(x) = \Delta_{E/F}(x) \cdot \chi_E(x) = \Delta_{E/F}(x) \cdot \chi_K(N_{K/E}^{-1}(x)),$$

since  $F^\times \subseteq \mathcal{N}_{K/E}$ , and  $\chi_E|_{\mathcal{N}_{K/E}} = \chi_K \circ N_{K/E}^{-1}$ .

---

<sup>2</sup>This condition  $F^\times \subseteq \mathcal{N}_{K/E}$  implies that for every  $x \in F^\times$  must have a preimage under the  $N_{K/E}$ , but the preimage is not unique.

Furthermore, since  $E/F$  is an abelian extension,  $\text{Gal}(E/F) \cong \widehat{\text{Gal}(E/F)}$ , and from Miller's Theorem (cf. [22], Theorem 6), we can write

$$\begin{aligned} \Delta_{E/F} &= \det(\text{Ind}_{E/F}(1)) \\ &= \det\left(\sum_{\chi \in \widehat{\text{Gal}(E/F)}} \chi\right) \\ &= \prod_{\chi \in \widehat{\text{Gal}(E/F)}} \chi \\ &= \begin{cases} 1 & \text{when } 2\text{-rank } \text{rk}_2(\text{Gal}(E/F)) \neq 1 \\ \omega_{E'/F}(x) & \text{when } 2\text{-rank } \text{rk}_2(\text{Gal}(E/F)) = 1, \end{cases} \end{aligned}$$

where  $E'/F$  is a uniquely determined quadratic subextension in  $E/F$ , and  $\omega_{E'/F}$  is the character of  $F^\times$  which corresponds to  $E'/F$  by class field theory.  $\square$

**3.1. Heisenberg representations of  $G_F$  of dimensions prime to  $p$ .** Let  $F/\mathbb{Q}_p$  be a non-archimedean local field, and  $G_F$  be the absolute Galois group of  $F$ . In this subsection we construct all Heisenberg representations of  $G_F$  of dimensions prime to  $p$ . Studying the construction of this type (i.e., dimension prime to  $p$ ) Heisenberg representations are important for our next section.

**Definition 3.8 (U-isotropic).** Let  $F$  be a non-archimedean local field. Let  $X : FF^\times \rightarrow \mathbb{C}^\times$  be an alternating character with the property

$$X(\varepsilon_1, \varepsilon_2) = 1, \quad \text{for all } \varepsilon_1, \varepsilon_2 \in U_F.$$

In other words,  $X$  is a character of  $FF^\times/U_F \wedge U_F$ . Then  $X$  is said to be the U-isotropic. These  $X$  are easy to classify:

**Lemma 3.9.** Fix a uniformizer  $\pi_F$  and write  $U := U_F$ . Then we obtain an isomorphism

$$\widehat{U} \cong FF^\times / U \wedge U, \quad \eta \mapsto X_\eta, \quad \eta_X \leftarrow X$$

between characters of  $U$  and U-isotropic alternating characters as follows:

$$(3.19) \quad X_\eta(\pi_F^a \varepsilon_1, \pi_F^b \varepsilon_2) := \eta(\varepsilon_1)^b \cdot \eta(\varepsilon_2)^{-a}, \quad \eta_X(\varepsilon) := X(\varepsilon, \pi_F),$$

where  $a, b \in \mathbb{Z}$ ,  $\varepsilon, \varepsilon_1, \varepsilon_2 \in U$ , and  $\eta : U \rightarrow \mathbb{C}^\times$ . Then

$$\text{Rad}(X_\eta) = \langle \pi_F^{\#\eta} \rangle \times \text{Ker}(\eta) = \langle (\pi_F \varepsilon)^{\#\eta} \rangle \times \text{Ker}(\eta),$$

does not depend on the choice of  $\pi_F$ , where  $\#\eta$  is the order of the character  $\eta$ , hence

$$F^\times / \text{Rad}(X_\eta) \cong \langle \pi_F \rangle / \langle \pi_F^{\#\eta} \rangle \times U / \text{Ker}(\eta) \cong \mathbb{Z}_{\#\eta} \times \mathbb{Z}_{\#\eta}.$$

Therefore all Heisenberg representations of type  $\rho = \rho(X_\eta, \chi)$  have dimension  $\dim(\rho) = \#\eta$ .

*Proof.* To prove  $\widehat{U} \cong FF^\times/\widehat{U} \wedge U$ , we have to show that  $\eta_{X_\eta} = \eta$  and  $X_{\eta_X} = X_\eta$ , and that the inverse map  $X \mapsto \eta_X$  does not depend on the choice of  $\pi_F$ .

From the above definition of  $\eta_X$ , we can write:

$$\eta_{X_\eta}(\varepsilon) = X_\eta(\varepsilon, \pi_F) = \eta(\varepsilon)^1 \cdot \eta(1)^0 = \eta(\varepsilon),$$

for all  $\varepsilon \in U$ , hence  $\eta_{X_\eta} = \eta$ .

Similarly, from the above definition of  $X$ , we have:

$$\begin{aligned} X_{\eta_X}(\pi_F^a \varepsilon_1, \pi_F^b \varepsilon) &= \eta_X(\varepsilon_1)^b \cdot \eta_X(\varepsilon_2)^{-a} = X(\varepsilon_1, \pi_F)^b \cdot X(\varepsilon_2, \pi_F)^{-a} \\ &= X(\varepsilon_1, \pi_F)^b \cdot X(\pi_F, \varepsilon_2)^a = X(\varepsilon_1, \pi_F^b) \cdot X(\pi_F^a, \varepsilon_2) \\ &= X(\pi_F^a \varepsilon_1, \pi_F^b \varepsilon). \end{aligned}$$

This shows that  $X_{\eta_X} = X$ .

Now we choose a uniformizer  $\pi_F \varepsilon$ , where  $\varepsilon \in U$ , instead of choosing  $\pi_F$ . Then we can write

$$\begin{aligned} X_\eta((\pi_F \varepsilon)^a \varepsilon_1, (\pi_F \varepsilon)^b \varepsilon_2) &= X_\eta(\pi_F^a (\varepsilon^a \varepsilon_1), \pi_F^b (\varepsilon^b \varepsilon_2)) \\ &= \eta(\varepsilon^a \varepsilon_1)^b \cdot \eta(\varepsilon^b \varepsilon_2)^{-a} \\ &= \eta(\varepsilon_1)^b \cdot \eta(\varepsilon_2)^{-a} \cdot \eta(\varepsilon^{ab-ab}) \\ &= \eta(\varepsilon_1)^b \cdot \eta(\varepsilon_2)^{-a} = X(\pi_F^a \varepsilon_1, \pi_F^b \varepsilon_2). \end{aligned}$$

This shows that  $X_\eta$  does not depend on the choice of the uniformizer  $\pi_F$ . Similarly since  $\eta_X(\varepsilon) := X(\varepsilon, \pi_F)$ , it is clear that  $\eta_X$  is also does not depend on the choice of the uniformizer  $\pi_F$ .

By the definition of the radical of  $X_\eta$ , we have:

$$\text{Rad}(X_\eta) = \{\pi_F^a \varepsilon \in F^\times \mid X_\eta(\pi_F^a \varepsilon, \pi_F^b \varepsilon') = \eta(\varepsilon)^b \cdot \eta(\varepsilon')^{-a} = 1\},$$

for all  $b \in \mathbb{Z}$ , and  $\varepsilon' \in U$ .

Now if we fix a uniformizer  $\pi_F \varepsilon''$ , where  $\varepsilon'' \in U$  instead of  $\pi_F$ , we can write:

$$\text{Rad}(X_\eta) = \{(\pi_F \varepsilon'')^a \varepsilon \in F^\times \mid X_\eta((\pi_F \varepsilon'')^a \varepsilon, (\pi_F \varepsilon'')^b \varepsilon') = \eta(\varepsilon''^a \varepsilon)^b \cdot \eta(\varepsilon''^b \varepsilon')^{-a} = \eta(\varepsilon)^b \cdot \eta(\varepsilon')^{-a} = 1\},$$

This gives  $\text{Rad}(X_\eta) = \langle \pi_F^{\# \eta} \rangle \times \text{Ker}(\eta) = \langle (\pi_F \varepsilon)^{\# \eta} \rangle \times \text{Ker}(\eta)$ , hence

$$F^\times / \text{Rad}(X_\eta) \cong \langle \pi_F \rangle / \langle \pi_F^{\# \eta} \rangle \times U / \text{Ker}(\eta) \cong \mathbb{Z}_{\# \eta} \times \mathbb{Z}_{\# \eta}.$$

Then all Heisenberg representations of type  $\rho = \rho(X_\eta, \chi)$  have dimension

$$\dim(\rho) = \sqrt{[F^\times : \text{Rad}(X_\eta)]} = \# \eta.$$

□

From the above Lemma 3.9 we know that the dimension of a U-isotropic Heisenberg representation  $\rho = \rho(X_\eta, \chi)$  of  $G_F$  is  $\dim(\rho) = \# \eta$ , and  $F^\times / \text{Rad}(X_\eta) \cong \mathbb{Z}_{\# \eta} \times \mathbb{Z}_{\# \eta}$ , a direct product of two cyclic (bicyclic) groups of the same order  $\# \eta$ . In general, if  $A = \mathbb{Z}_m \times \mathbb{Z}_m$  is a bicyclic group of order  $m^2$ , then by the following lemma we can compute total number of elements of order  $m$  in  $A$ , and number of cyclic complementary subgroup of a fixed cyclic subgroup of order  $m$ .

**Lemma 3.10.** *Let  $A \cong \mathbb{Z}_m \times \mathbb{Z}_m$  be a bicyclic abelian group of order  $m^2$ . Then:*

- (1) Then number  $\psi(m)$  of cyclic subgroups  $B \subset A$  of order  $m$  is a multiplicative arithmetic function (i.e.,  $\psi(mn) = \psi(m)\psi(n)$  if  $\gcd(m, n) = 1$ ).
- (2) Explicitly we have

$$(3.20) \quad \psi(m) = m \cdot \prod_{p|m} \left(1 + \frac{1}{p}\right).$$

And the number of elements of order  $m$  in  $A$  is:

$$(3.21) \quad \varphi(m) \cdot \psi(m) = m^2 \cdot \prod_{p|m} \left(1 - \frac{1}{p^2}\right).$$

Here  $p$  is a prime divisor of  $m$  and  $\varphi(n)$  is the Euler's totient function of  $n$ .

- (3) Let  $B \subset A$  be cyclic of order  $m$ . Then  $B$  has always a complementary subgroup  $B' \subset A$  such that  $A = B \times B'$ , and  $B'$  is again cyclic of order  $m$ . And for  $B$  fixed, the number of all different complementary subgroups  $B'$  is  $= m$ .

*Proof.* To prove these assertions we need to recall the fact: If  $G$  is a finite cyclic group of order  $m$ , then number of generators of  $G$  is  $\varphi(m) = m \prod_{p|m} \left(1 - \frac{1}{p}\right)$ .

(1). By the given condition  $A \cong \mathbb{Z}_m \times \mathbb{Z}_m$  and  $\psi(m)$  is the number of cyclic subgroup of  $A$  of order  $m$ . Then it is clear that  $\psi$  is an arithmetic function with  $\psi(1) = 1 \neq 0$ , hence  $\psi$  is not **additive**. Now take  $m \geq 2$ , and the prime factorization of  $m$  is:  $m = \prod_{i=1}^k p_i^{a_i}$ . To prove this, first we should start with  $m = p^n$ , hence  $A \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n}$ . Then number of subgroup of  $A$  of order  $p^n$  is:

$$\psi(p^n) = \frac{2\varphi(p^n)p^n - \varphi(p^n)^2}{\varphi(p^n)} = 2p^n - \varphi(p^n) = p^n(2 - 1 + \frac{1}{p}) = p^n(1 + \frac{1}{p}).$$

Now take  $m = p^n q^r$ , where  $p, q$  are both prime with  $\gcd(p, q) = 1$ . We also know that  $\mathbb{Z}_{p^n q^r} \times \mathbb{Z}_{p^n q^r} \cong \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n} \times \mathbb{Z}_{q^r} \times \mathbb{Z}_{q^r}$ . This gives  $\psi(p^n q^r) = \psi(p^n) \cdot \psi(q^r)$ . By the similar method we can show that  $\psi(m) = \prod_{i=1}^k \psi(p_i^{a_i})$ , where  $m = \prod_{i=1}^k p_i^{a_i}$ . This condition implies that  $\psi$  is a multiplicative arithmetic function.

- (2). Since  $\psi$  is multiplicative arithmetic function, we have

$$\begin{aligned} \psi(m) &= \prod_{i=1}^k \psi(p_i^{a_i}) = \prod_{i=1}^k p_i^{a_i} \left(1 + \frac{1}{p_i}\right) \quad \text{since } \psi(p^n) = p^n \left(1 + \frac{1}{p}\right), \\ &= p_1^{a_1} \cdots p_k^{a_k} \prod_{i=1}^k \left(1 + \frac{1}{p_i}\right) = m \cdot \prod_{p|m} \left(1 + \frac{1}{p}\right). \end{aligned}$$

We also know that number of generator of a finite cyclic group of order  $m$  is  $\varphi(m)$ , hence number of elements of order  $m$  is  $\varphi(m)$ . Then the number of elements of order  $m$  in  $A$  is:

$$\varphi(m) \cdot \psi(m) = m \cdot \prod_{p|m} \left(1 - \frac{1}{p}\right) \cdot m \prod_{p|m} \left(1 + \frac{1}{p}\right) = m^2 \cdot \prod_{p|m} \left(1 - \frac{1}{p^2}\right).$$

- (3). Let  $B \subset A$  be a cyclic subgroup of order  $m$ . Since  $A$  is abelian and bicyclic of order  $m^2$ ,  $B$  has always a complementary subgroup  $B' \subset A$  such that  $A = B \times B'$ , and  $B'$  is again cyclic (because  $A$  is cyclic, hence  $A/B$  and  $|A/B| = m$ ) of order  $m$ .

To prove the last part of (3), we start with  $m = p^n$ . Here  $B$  is a cyclic subgroup of  $A$  of order  $p^n$ , hence  $B = \langle (a, e) \rangle$ , where  $\#a = p^n$ , and  $e$  is the identity of  $B'$ . Since  $B$  has complementary cyclic subgroup, namely  $B'$ , of order  $p^n$ . we can choose  $B' = \langle (b, c) \rangle$ , where  $B \cap B' = (e, e)$ . This gives that  $c$  is a generator of  $B'$ , and  $b$  could be any element in  $\mathbb{Z}_{p^n}$ . Thus total number  $\psi_{B'}(p^n)$  of all different complementary subgroups  $B'$  is:

$$\psi_{B'}(p^n) = \frac{p^n \varphi(p^n)}{\varphi(p^n)} = p^n = m.$$

Now if we take  $m = p^n q^r$ , where  $q$  is a different prime from  $p$ . Then by same method we can see that  $\psi_{B'}(p^n q^r) = \psi_{B'}(p^n) \cdot \psi_{B'}(q^r) = p^n q^r = m$ . Thus for arbitrary  $m$  we can conclude that  $\psi_{B'}(m) = m$ . □

In the following lemma, we give an equivalent condition for U-isotropic Heisenberg representation.

**Lemma 3.11.** *Let  $G_F$  be the absolute Galois group of a non-archimedean local field  $F$ . For a Heisenberg representation  $\rho = \rho(Z, \chi_\rho) = \rho(X, \chi_K)$  the following are equivalent:*

- (1) *The alternating character  $X$  is U-isotropic.*
- (2) *Let  $E/F$  be the maximal unramified subextension in  $K/F$ . Then  $\text{Gal}(K/E)$  is maximal isotropic for  $X$ .*
- (3)  *$\rho = \text{Ind}_{E/F}(\chi_E)$  can be induced from a character  $\chi_E$  of  $E^\times$  (where  $E$  is as in (2)).*

*Proof.* This proof follows from the above Lemma 3.9.

First, assume that  $X$  is U-isotropic, i.e.,  $X \in \widehat{FF^\times/U} \wedge U$ . We also know that  $\widehat{U} \cong \widehat{FF^\times/U} \wedge U$ . Then  $X$  corresponds a character of  $U$ , namely  $X \mapsto \eta_X$ . Then from Lemma 3.9 we have  $F^\times/\text{Rad}(X) \cong \mathbb{Z}_{\#\eta_X} \times \mathbb{Z}_{\#\eta_X}$ , i.e., product of two cyclic groups of same order.

Since  $K/F$  is the abelian bicyclic extension which corresponds to  $\text{Rad}(X)$ , we can write:

$$\mathcal{N}_{K/F} = \text{Rad}(X), \quad \text{Gal}(K/F) \cong F^\times/\text{Rad}(X).$$

Let  $E/F$  be the maximal unramified subextension in  $K/F$ . Then  $[E : F] = \#\eta_K$  because the order of maximal cyclic subgroup of  $\text{Gal}(K/F)$  is  $\#\eta_X$ . Then  $f_{E/F} = \#\eta_X$ , hence  $f_{K/F} = e_{K/F} = \#\eta_X$  because  $f_{K/F} \cdot e_{K/F} = [K : F] = \#\eta_X^2$  and  $\text{Gal}(K/F)$  is not cyclic group.

Now we have to prove that the extension  $E/F$  corresponds to a maximal isotropic for  $X$ . Let  $H/Z$  be a maximal isotropic for  $X$ , hence  $[G_F/Z : H/Z] = \#\eta_X$ , hence  $H/Z = \text{Gal}(K/E)$ , i.e., the maximal unramified subextension  $E/F$  in  $K/F$  corresponds to a maximal isotropic subgroup, hence

$$\rho(X, \chi_K) = \text{Ind}_{E/F}(\chi_E), \text{ for } \chi_E \circ N_{K/E} = \chi_K.$$

Finally, since  $E/F$  is unramified and the extension  $E$  corresponds a maximal isotropic subgroup for  $X$ , we have  $U_F \subset \mathcal{N}_{E/F}$ , hence  $U_F \subset \mathcal{N}_{K/F}$  and  $X|_{U \times U} = 1$  because  $U_F \subset F^\times \subset \mathcal{N}_{K/E}$ . This shows that  $X$  is U-isotropic. □

**Corollary 3.12.** *The U-isotropic Heisenberg representation  $\rho = \rho(X_\eta, \chi)$  can never be wild because it is induced from an unramified extension  $E/F$ , but the dimension  $\dim(\rho(X_\eta, \chi)) =$*



$\# \eta$  can be a power of  $p$ .

The representations  $\rho$  of dimension prime to  $p$  are precisely given as  $\rho = \rho(X_\eta, \chi)$  for characters  $\eta$  of  $U/U^1$ .

*Proof.* This is clear from the above lemma 3.9 and the fact  $|U/U^1| = q_F - 1$ .  $\square$

**Remark 3.13 (Arithmetic description of representations  $\rho(X_\eta, \chi)$ ):** We let  $K_\eta|F$  be the abelian bicyclic extension which corresponds to  $\text{Rad}(X_\eta)$  :

$$\mathcal{N}_{K_\eta/F} = \text{Rad}(X_\eta), \quad \text{Gal}(K_\eta/F) \cong F^\times / \text{Rad}(X_\eta).$$

Then we have  $f_{K_\eta|F} = e_{K_\eta|F} = \# \eta$  and the maximal unramified subextension  $E/F \subset K_\eta/F$  corresponds to a maximal isotropic subgroup, hence

$$\rho(X_\eta, \chi) = \text{Ind}_{E/F}(\chi_E), \quad \text{for } \chi_E \circ N_{K_\eta/E} = \chi.$$

We recall here that  $\chi : K_\eta^\times / I_F K_\eta^\times \rightarrow \mathbb{C}^\times$  is a character such that (cf. Theorem 3.1(3))

$$\chi|_{(K_\eta^\times)_F} \leftrightarrow X_\eta, \quad \text{with respect to } (K_\eta^\times)_F / I_F K_\eta^\times \cong F^\times / \text{Rad}(X_\eta) \wedge F^\times / \text{Rad}(X_\eta).$$

In particular, we see that  $(K_\eta^\times)_F / I_F K_\eta^\times$  is cyclic of order  $\# \eta$  and  $\chi|_{(K_\eta^\times)_F}$  must be a faithful character of that cyclic group.

In the following lemma we see the explicit description of the representation  $\rho = \rho(X_\eta, \chi)$ .

**Lemma 3.14 (Explicit Lemma).** *Let  $\rho = \rho(X_\eta, \chi_K)$  be a  $U$ -isotropic Heisenberg representation of the absolute Galois group  $G_F$  of a local field  $F/\mathbb{Q}_p$ . Let  $K = K_\eta$  and let  $E/F$  be the maximal unramified subextension in  $K/F$ . Then:*

(1) *The norm map induces an isomorphism:*

$$N_{K/E} : K_F^\times / I_F K^\times \xrightarrow{\sim} I_F E^\times / I_F \mathcal{N}_{K/E}.$$

(2) *Let  $c_{K/F} : F^\times / \text{Rad}(X_\eta) \wedge F^\times / \text{Rad}(X_\eta) \cong K_F^\times / I_F K^\times$  be the isomorphism which is induced by the commutator in the relative Weil-group  $W_{K/F}$ . Then for units  $\varepsilon \in U_F$  we explicitly have:*

$$c_{K/F}(\varepsilon \wedge \pi_F) = N_{K/E}^{-1}(N_{E/F}^{-1}(\varepsilon)^{1-\varphi_{E/F}}),$$

where  $\varphi_{E/F}$  is the Frobenius automorphism for  $E/F$  and where  $N^{-1}$  means to take a preimage of the norm map.

(3) *The restriction  $\chi_K|_{K_F^\times}$  is characterized by:*

$$\chi_K \circ c_{K/F}(\varepsilon \wedge \pi_F) = X_\eta(\varepsilon, \pi_F) = \eta(\varepsilon),$$

for all  $\varepsilon \in U_F$ , where  $c_{K/F}(\varepsilon \wedge \pi_F)$  is explicitly given via (2).

*Proof. (1).* By the given conditions we have:  $K = K_\eta$ , and  $K/F$  is the bicyclic extension with  $\text{Rad}(X_\eta) = \mathcal{N}_{K/F}$ , and  $E/F$  is the maximal unramified subextension in  $K/F$ . So  $K/E$  and  $E/F$  both are cyclic, hence

$$E_F^\times = I_F E^\times, \quad K_E^\times = I_E K^\times.$$

From the diagram (3.6.1) on p. 41 of [7], we have

$$N_{K/E} : K_F^\times / I_F K^\times \xrightarrow{\sim} E_F^\times / I_F \mathcal{N}_{K/E}.$$

We also know that  $E_F^\times = I_F E^\times$ . Thus the norm map  $N_{K/E}$  induces an isomorphism:

$$N_{K/E} : K_F^\times / I_F K^\times \cong I_F E^\times / I_F \mathcal{N}_{K/E}.$$

(2). By the given conditions,  $c_{K/F}$  is the isomorphism which is induced by the commutator in the relative Weil-group  $W_{K/F}$  (cf. the map (3.3). Here  $\text{Rad}(X_\eta) = \mathcal{N}_{K/F} =: N$ . Then from Proposition 1(iii) of [8] on p. 128, we have

$$c_{K/F} : N \wedge F^\times / N \wedge N \xrightarrow{\sim} I_F K^\times / I_F K_F^\times$$

as an isomorphism by the map:

$$c_{K/F}(x \wedge y) = N_{K/F}^{-1}(x)^{1-\phi_F(y)},$$

where  $\phi_F(y) \in \text{Gal}(K/F)$  for  $y \in F^\times$  by class field theory. If  $y = \pi_F$ , then by class field theory (cf. [18], p. 20, Theorem 1.1(a)), we can write  $\phi_F(\pi_F)|_E = \varphi_{E/F}$ , where  $\varphi_{E/F}$  is the Frobenius automorphism for  $E/F$ .

Now we come to our special case. Since  $E/F$  is unramified, we have  $U_F \subset \mathcal{N}_{E/F}$ , and we obtain (cf. [7], pp. 46-47 of Section 4.4 and the diagram on p. 302 of [5]):

$$(3.22) \quad N_{K/E} \circ c_{K/F}(\varepsilon \wedge \pi_F) = N_{E/F}^{-1}(\varepsilon)^{1-\varphi_{E/F}}.$$

We also know (see the first two lines under the upper diagram on p. 302 of [5]) that  $E_F^\times \subseteq \mathcal{N}_{K/E}$ . Here

$$N_{E/F}^{-1}(\varepsilon)^{1-\varphi_{E/F}} \in I_F E^\times / I_F \mathcal{N}_{K/E} = E_F^\times / I_F \mathcal{N}_{K/E},$$

because  $E/F$  is cyclic, hence  $E_F^\times = I_F E^\times$ . Therefore from equation (3.22) we can conclude:

$$c_{K/F}(\varepsilon \wedge \pi_F) = N_{K/E}^{-1}(N_{E/F}^{-1}(\varepsilon)^{1-\varphi_{E/F}}).$$

(3.) We know that the  $c_{K/F}(\varepsilon \wedge \pi_F) \in K_F^\times$  and  $\chi_K : K^\times / I_F K^\times \rightarrow \mathbb{C}^\times$ . Then we can write

$$\begin{aligned} \chi_K \circ c_{K/F}(\varepsilon \wedge \pi_F) &= \chi_K(N_{K/E}^{-1}(N_{E/F}^{-1}(\varepsilon)^{1-\varphi_{E/F}})) \\ &= \chi_E \circ N_{K/E}(N_{K/E}^{-1}(N_{E/F}^{-1}(\varepsilon)^{1-\varphi_{E/F}})), \quad \text{since } \chi_K = \chi_E \circ N_{K/E} \\ &= \chi_E(N_{E/F}^{-1}(\varepsilon)^{1-\varphi_{E/F}}) = X_\eta(\varepsilon, \pi_F) \\ &= \eta(\varepsilon). \end{aligned}$$

This is true for all  $\varepsilon \in U_F$ . Therefore we can conclude that  $\chi_K|_{K_F^\times} = \eta$ . □

**Example 3.15 (Explicit description of Heisenberg representations of dimension prime to  $p$ ).** Let  $F/\mathbb{Q}_p$  be a local field, and  $G_F$  be the absolute Galois group of  $F$ . Let  $\rho = \rho(X, \chi_K)$  be a Heisenberg representation of  $G_F$  of dimension  $m$  prime to  $p$ . Then from Corollary 3.12 the alternating character  $X = X_\eta$  is  $U$ -isotropic for a character  $\eta : U_F/U_F^1 \rightarrow \mathbb{C}^\times$ . Here from Lemma 3.9 we can say  $m = \sqrt{[F^\times : \text{Rad}(X_\eta)]} = \#\eta$  divides  $q_F - 1$ .

Since  $U_F^1$  is a pro- $p$ -group and  $\gcd(m, p) = 1$ , we have  $(U_F^1)^m = U_F^1 \subset F^{\times m}$ , and therefore

$$F^\times / F^{\times m} \cong \mathbb{Z}_m \times \mathbb{Z}_m,$$

is a bicyclic group of order  $m^2$ . So by class field theory there is precisely one extension  $K/F$  such that  $\text{Gal}(K/F) \cong \mathbb{Z}_m \times \mathbb{Z}_m$  and the norm group  $\mathcal{N}_{K/F} := N_{K/F}(K^\times) = F^{\times m}$ .

We know that  $U_F/U_F^1$  is a cyclic group of order  $q_F - 1$ , hence  $\widehat{U_F/U_F^1} \cong U_F/U_F^1$ . By the given condition  $m|(q_F - 1)$ , hence  $U_F/U_F^1$  has exactly one subgroup of order  $m$ . Then number of elements of order  $m$  in  $U_F/U_F^1$  is  $\varphi(m)$ , the Euler's  $\varphi$ -function of  $m$ . In this setting, we have  $\eta \in \widehat{U_F/U_F^1} \cong \widehat{FF^\times/U_F^1} \wedge U_F^1$  with  $\#\eta = m$ . This implies that up to 1-dimensional character twist there are  $\varphi(m)$  representations corresponding to  $X_\eta$  where  $\eta : U_F/U_F^1 \rightarrow \mathbb{C}^\times$  is of order  $m$ . According to Corollary 1.2 of [5], all dimension- $m$ -Heisenberg representations of  $G_F = \text{Gal}(\overline{F}/F)$  are given as

$$(1H) \quad \rho = \rho(X_\eta, \chi_K),$$

where  $\chi_K : K^\times/I_F K^\times \rightarrow \mathbb{C}^\times$  is a character such that the restriction of  $\chi_K$  to the subgroup  $K_F^\times$  corresponds to  $X_\eta$  under the map (3.3), and

$$(2H) \quad F^\times/F^{\times m} \wedge F^\times/F^{\times m} \cong K_F^\times/I_F K^\times,$$

which is given via the commutator in the relative Weil-group  $W_{K/F}$  (for details arithmetic description of Heisenberg representations of a Galois group, see [5], pp. 301-304). The condition (2H) corresponds to (3.3). Here the above Explicit Lemma 3.14 comes in.

Here due to our assumption both sides of (2H) are groups of order  $m$ . And if one choice  $\chi_K = \chi_0$  has been fixed, then all other  $\chi_K$  are given as

$$(3.23) \quad \chi_K = (\chi_F \circ N_{K/F}) \cdot \chi_0,$$

for arbitrary characters of  $F^\times$ . For an optimal choice  $\chi_K = \chi_0$ , and order of  $\chi_0$  we need the following lemma.

**Lemma 3.16.** *Let  $K/F$  be the extension of  $F/\mathbb{Q}_p$  for which  $\text{Gal}(K/F) = \mathbb{Z}_m \times \mathbb{Z}_m$ . The  $K_F^\times$  and  $I_F K^\times$  are as above. Then the sequence*

$$(3.24) \quad 1 \rightarrow U_K^1 K_F^\times / U_K^1 I_F K^\times \rightarrow U_K / U_K^1 I_F K^\times \xrightarrow{N_{K/F}} U_F / U_F^1 \rightarrow U_F / U_F \cap F^{\times m} \rightarrow 1$$

*is exact, and the outer terms are both of order  $m$ , hence inner terms are both cyclic of order  $q_F - 1$ .*

*Proof.* The sequence is exact because  $F^{\times m} = N_{K/F}(K^\times)$  is the group of norms, and  $F^\times/F^{\times m} \cong \mathbb{Z}_m \times \mathbb{Z}_m$  implies that the right hand term<sup>3</sup> is of order  $m$ . By our assumption the order of  $K_F^\times/I_F K^\times$  is  $m$ . Now we consider the exact sequence

$$(3.25) \quad 1 \rightarrow U_K^1 \cap K_F^\times / U_K^1 \cap I_F K^\times \rightarrow K_F^\times / I_F K^\times \rightarrow U_K^1 K_F^\times / U_K^1 I_F K^\times \rightarrow 1.$$

<sup>3</sup>Since  $\gcd(m, p) = 1$ , we have

$$U_F \cdot F^{\times m} = \langle \zeta \rangle \times U_F^1 (\langle \pi_F^m \rangle \times \langle \zeta^m \rangle \times U_F^1) = \langle \pi_F^m \rangle \times \langle \zeta \rangle \times U_F^1,$$

where  $\zeta$  is a  $(q_F - 1)$ -st root of unity. Then

$$U_F / U_F \cap F^{\times m} = U_F \cdot F^{\times m} / F^{\times m} = \langle \pi_F^m \rangle \times \langle \zeta \rangle \times U_F^1 / \langle \pi_F^m \rangle \times \langle \zeta^m \rangle \times U_F^1 \cong \mathbb{Z}_m.$$

Hence  $|U_F / U_F \cap F^{\times m}| = m$ .

Since the middle term has order  $m$ , the left term must have order 1, because  $U_K^1$  is a pro- $p$ -group and  $\gcd(m, p) = 1$ . Hence the right term is also of order  $m$ . So the outer terms of the sequence (3.24) have both order  $m$ , hence the inner terms must have the same order  $q_F - 1 = [U_F : U_F^1]$ , and they are cyclic, because the groups  $U_F/U_F^1$  and  $U_K/U_K^1$  are both cyclic.  $\square$

**We now are in a position to choose  $\chi_K = \chi_0$  as follows:**

- (1) we take  $\chi_0$  as a character of  $K^\times/U_K^1 I_F K^\times$ ,
- (2) we take it on  $U_K^1 K_F^\times/U_K^1 I_F K^\times$  as it is prescribed by the above Explicit Lemma 3.14, in particular,  $\chi_0$  restricted to that subgroup (which is cyclic of order  $m$ ) will be faithful.
- (3) we take it trivial on all primary components of the cyclic group  $U_K/U_K^1 I_F K^\times$  which are not  $p_i$ -primary, where  $m = \prod_{i=1}^n p_i^{a_i}$ .
- (4) we take it trivial for a fixed prime element  $\pi_K$ .

Under the above optimal choice of  $\chi_0$ , we have

**Lemma 3.17.** *Denote  $\nu_p(n) :=$  as the highest power of  $p$  for which  $p^{\nu_p(n)} | n$ . The character  $\chi_0$  must be a character of order*

$$m_{q_F-1} := \prod_{l|m} l^{\nu_l(q_F-1)},$$

which we will call the  $m$ -primary part of  $q_F - 1$ , so it determines a cyclic extension  $L/K$  of degree  $m_{q_F-1}$  which is totally tamely ramified, and we can consider the Heisenberg representation  $\rho = (X, \chi_0)$  of  $G_F = \text{Gal}(\bar{F}/F)$  is a representation of  $\text{Gal}(L/F)$ , which is of order  $m^2 \cdot m_{q_F-1}$ .

*Proof.* By the given conditions,  $m | q_F - 1$ . Therefore we can write

$$q_F - 1 = \prod_{l|m} l^{\nu_l(q_F-1)} \cdot \prod_{p|q_F-1, p \nmid m} p^{\nu_p(q_F-1)} = m_{q_F-1} \cdot \prod_{p|q_F-1, p \nmid m} p^{\nu_p(q_F-1)},$$

where  $l, p$  are prime, and  $m_{q_F-1} = \prod_{l|m} l^{\nu_l(q_F-1)}$ .

From the construction of  $\chi_0$ ,  $\pi_K \in \text{Ker}(\chi_0)$ , hence the order of  $\chi_0$  comes from the restriction to  $U_K$ . Then the order of  $\chi_0$  is  $m_{q_F-1}$ , because from Lemma 3.16, the order of  $U_K/U_K^1 I_F K$  is  $q_F - 1$ . Since order of  $\chi_0$  is  $m_{q_F-1}$ , by class field theory  $\chi_0$  determines a cyclic extension  $L/K$  of degree  $m_{q_F-1}$ , hence

$$N_{L/K}(L^\times) = \text{Ker}(\chi_0) = \text{Ker}(\rho).$$

This means  $G_L$  is the kernel of  $\rho(X, \chi_0)$ , hence  $\rho(X, \chi_0)$  is actually a representation of  $G_F/G_L \cong \text{Gal}(L/F)$ .

Since  $G_L$  is normal subgroup of  $G_F$ , hence  $L/F$  is a normal extension of degree  $[L : F] = [L : K] \cdot [K : F] = m_{q_F-1} \cdot m^2$ . Thus  $\text{Gal}(L/F)$  is of order  $m^2 \cdot m_{q_F-1}$ .

Moreover since  $[L : K] = m_{q_F-1}$  and  $\gcd(m, p) = 1$ ,  $L/K$  is tame. By construction we have a prime  $\pi_K \in \text{Ker}(\chi_0) = N_{L/K}(L^\times)$ , hence  $L/K$  is totally ramified extension.  $\square$

**Lemma 3.18.** (Here  $L$ ,  $K$ , and  $F$  are the same as in Lemma 3.17) Let  $F^{ab}/F$  be the maximal abelian extension. Then we have

$$L \supset L \cap F^{ab} \supset K \supset F, \quad \{1\} \subset G' \subset Z(G) \subset G = \text{Gal}(L/F),$$

where  $[L : L \cap F^{ab}] = |G| = m$  and  $[L : K] = |Z(G)| = m_{q_F-1}$ .

*Proof.* Let  $F^{ab}/F$  be the maximal abelian extension. Then we have

$$L \supset L \cap F^{ab} \supset K \supset F.$$

Here  $L \cap F^{ab}/F$  is the maximal abelian in  $L/F$ . Then from Galois theory we can conclude

$$\text{Gal}(L/L \cap F^{ab}) = [\text{Gal}(L/F), \text{Gal}(L/F)] =: G'.$$

Since  $\text{Gal}(L/F) = G_F/\text{Ker}(\rho)$ , and  $[[G_F, G_F], G_F] \subseteq \text{Ker}(\rho)$ , from relation (3.3) we have

$$G' = [G_F, G_F]/\text{Ker}(\rho) \cap [G_F, G_F] = [G_F, G_F]/[[G_F, G_F], G_F] \cong K_F^\times/I_F K^\times.$$

Again from sequence 3.25 we have  $|U_K^1 K_F^\times / U_K^1 I_F K^\times| = |K_F^\times / I_F K^\times| = m$ . Hence  $|G'| = m$ .

From the Heisenberg property of  $\rho$ , we have  $[[G_F, G_F], G_F] \subseteq \text{Ker}(\rho)$ , hence  $\text{Gal}(L/F) = G_F/\text{Ker}(\rho)$  is a two-step nilpotent group (cf. Remark 2.9). This gives  $[G', G] = 1$ , hence  $G' \subseteq Z := Z(G)$ . Thus  $G/Z$  is abelian.

Moreover, here  $Z$  is the scalar group of  $\rho$ , hence the dimension of  $\rho$  is:

$$\dim(\rho) = \sqrt{[G : Z]} = m$$

Therefore the order of  $Z$  is  $m_{q_F-1}$  and  $Z = \text{Gal}(L/K)$ .

□

*Remark 3.19 (Special case:  $m = 2$ , hence  $p \neq 2$ ).* Now if we take  $m = 2$ , hence  $p \neq 2$ , and choose  $\chi_0$  as the above optimal choice, then we will have  $m_{q_F-1} = 2_{q_F-1} = 2$ -primary factor of the number  $q_F - 1$ , and  $\text{Gal}(L/F)$  is a 2-group of order  $4 \cdot 2_{q_F-1}$ .

When  $q_F \equiv -1 \pmod{4}$ ,  $q_F$  is of the form  $q_F = 4l - 1$ , where  $l \geq 1$ . So we can write  $q_F - 1 = 2(2l - 1)$ . Since  $2l - 1$  is always odd, therefore when  $q_F \equiv -1 \pmod{4}$ , the order of  $\chi_0$  is  $2_{q_F-1} = 2$ . Then  $\text{Gal}(L/F)$  will be of order 8 if and only if  $q_F \equiv -1 \pmod{4}$ , i.e., if and only if  $i \notin F$ . And if  $q_F \equiv 1 \pmod{4}$ , then similarly, we can write  $q_F - 1 = 4m$  for some integer  $m \geq 1$ , hence  $2_{q_F-1} \geq 4$ . Therefore when  $q_F \equiv 1 \pmod{4}$ , the order of  $\text{Gal}(L/F)$  will be at least 16.

### 3.2. Artin conductors, Swan conductors, and the dimensions of Heisenberg representations.

**Definition 3.20 (Artin and Swan conductor).** Let  $G$  be a finite group and  $R(G)$  be the complex representation ring of  $G$ . For any two representations  $\rho_1, \rho_2 \in R(G)$  with characters  $\chi_1, \chi_2$  respectively, we have the Schur's inner product:

$$\langle \rho_1, \rho_2 \rangle_G = \langle \chi_1, \chi_2 \rangle_G := \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \cdot \overline{\chi_2(g)}.$$

Let  $K/F$  be a finite Galois group with Galois group  $G := \text{Gal}(K/F)$ . For an element  $g \in G$  different from identity 1, we define the positive integer (cf. [15], Chapter IV, p. 62)

$$i_G(g) := \inf\{\nu_K(x - g(x)) \mid x \in O_K\}.$$

By using this non-negative (when  $g \neq 1$ ) integer  $i_G(g)$  we define a function  $a_G : G \rightarrow \mathbb{Z}$  as follows:

$$a_G(g) = -f_{K/F} \cdot i_G(g) \text{ when } g \neq 1, \text{ and } a_G(1) = f_{K/F} \sum_{g \neq 1} i_G(g).$$

Thus from this definition we can see that  $\sum_{g \in G} a_G(g) = 0$ , hence  $\langle a_G, 1_G \rangle = 0$ . It can be proved (cf. [15], p. 99, Theorem 1) that the function  $a_G$  is the character of a linear representation of  $G$ , and that corresponding linear representation is called the **Artin representation**  $A_G$  of  $G$ .

Similarly, for a nontrivial  $g \neq 1 \in G$ , we define (cf. [29], p. 247)

$$s_G(g) = \inf\{\nu_K(1 - g(x)x^{-1}) \mid x \in K^\times\}, \quad s_G(1) = - \sum_{g \neq 1} s_G(g).$$

And we can define a function  $sw_G : G \rightarrow \mathbb{Z}$  as follows:

$$sw_G(g) = -f_{K/F} \cdot s_G(g)$$

It can also be shown that  $sw_G$  is a character of a linear representation of  $G$ , and that corresponding representation is called the **Swan representation**  $SW_G$  of  $G$ .

From [16], p. 160, we have the relation between the Artin and Swan representations (cf. [29], p. 248, equation (6.1.9))

$$(3.26) \quad SW_G = A_G + \text{Ind}_{G_0}^G(1) - \text{Ind}_{\{1\}}^G(1),$$

$G_0$  is the 0-th ramification group (i.e., inertia group) of  $G$ .

Now we are in a position to define the Artin and Swan conductor of a representation  $\rho \in R(G)$ . The Artin conductor of a representation  $\rho \in R(G)$  is defined by

$$a_F(\rho) := \langle A_G, \rho \rangle_G = \langle a_G, \chi \rangle_G,$$

where  $\chi$  is the character of the representation  $\rho$ . Similarly, for the representation  $\rho$ , the Swan conductor is:

$$sw_F(\rho) := \langle SW_G, \rho \rangle_G = \langle sw_G, \chi \rangle_G.$$

For more details about Artin and Swan conductor, see Chapter 6 of [29] and Chapter VI of [15].

From equation (3.26) we obtain

$$(3.27) \quad a_F(\rho) = sw_F(\rho) + \dim(\rho) - \langle 1, \rho \rangle_{G_0}.$$

Moreover, from Corollary of Proposition 4 on p. 101 of [15], for an induced representation  $\rho := \text{Ind}_{\text{Gal}(K/E)}^{\text{Gal}(K/F)}(\rho_E) = \text{Ind}_{E/F}(\rho_E)$ , we have

$$(3.28) \quad a_F(\rho) = f_{E/F} \cdot (d_{E/F} \cdot \dim(\rho_E) + a_E(\rho_E)).$$

We apply this formula (3.28) for  $\rho_E = \chi_E$  of dimension 1 and then conversely

$$a(\chi_E) = \frac{a_F(\rho)}{f_{E/F}} - d_{E/F}.$$

So if we know  $a_F(\rho)$  then we can compute  $a(\chi_E)$ .

Let  $\{G^i\}$ , where  $i \geq 0$ ,  $\in \mathbb{Q}$  be the ramification subgroups (in the upper numbering) of a local Galois group  $G$ . Now let  $\rho$  be an irreducible representation of  $G$ . For this irreducible  $\rho$  we define

$$j(\rho) := \max\{i \mid \rho|_{G^i} \neq 1\}.$$

Now if  $\rho$  is an irreducible representation of  $G$ , then  $\rho|_I \neq 1$ , where  $I = G^0 = G_0$  is the inertia subgroup of  $G$ . Thus from the definition of  $j(\rho)$  we can say, if  $\rho$  is irreducible, then we always have  $j(\rho) \geq 0$ , i.e.,  $\rho$  is nontrivial on the inertia group  $G_0$ . Then from the definitions of Swan and Artin conductors, and equation (3.27), when  $\rho$  is irreducible, we have the following relations

$$(3.29) \quad \text{sw}_F(\rho) = \dim(\rho) \cdot j(\rho), \quad a_F(\rho) = \dim(\rho) \cdot (j(\rho) + 1).$$

From the Theorem of Hasse-Arf (cf. [15], p. 76), if  $\dim(\rho) = 1$ , i.e.,  $\rho$  is a character of  $G/[G, G]$ , we can say that  $j(\rho)$  must be an integer, then  $\text{sw}_F(\rho) = j(\rho)$ ,  $a_F(\rho) = j(\rho) + 1$ . Moreover, by class field theory,  $\rho$  corresponds to a linear character  $\chi_F$ , hence for linear character  $\chi_F$ , we can write

$$j(\chi_F) := \max\{i \mid \chi_F|_{U_F^i} \neq 1\},$$

because under class field theory (under Artin isomorphism) the upper numbering in the filtration of  $\text{Gal}(F_{\text{ab}}/F)$  is compatible with the filtration (descending chain) of the group of units  $U_F$ .

From equation (3.29), it is easy to see that for higher dimensional  $\rho$ , we have  $\text{sw}_F(\rho), a_F(\rho)$  multiples of  $\dim(\rho)$  if and only if  $j(\rho)$  is an integer.

Now we come to our Heisenberg representations. For each  $X \in \widehat{FF^\times}$  we define

$$(3.30) \quad j(X) := \begin{cases} 0 & \text{when } X \text{ is trivial} \\ \max\{i \mid X|_{UU^i} \neq 1\} & \text{when } X \text{ is nontrivial,} \end{cases}$$

where  $UU^i \subseteq FF^\times$  is a subgroup which under (3.2) corresponds

$$G_F^i \cap [G_F, G_F]/G_F^i \cap [[G_F, G_F], G_F] \subseteq [G_F, G_F]/[[G_F, G_F], G_F].$$

Let  $\rho = \rho(X_\rho, \chi_K)$  be the **minimal conductor** (i.e., a representation with the smallest Artin conductor) Heisenberg representation for  $X_\rho$  of the absolute Galois group  $G_F$ . From Theorem 3 on p. 125 of [8], we have

$$(3.31) \quad \text{sw}_F(\rho) = \dim(\rho) \cdot j(X_\rho) = \sqrt{[F^\times : \text{Rad}(X_\rho)]} \cdot j(X_\rho).$$

Let  $\rho_0 = \rho_0(X, \chi_0)$  be a minimal representation corresponding  $X$ , then all other Heisenberg representations of dimension  $\dim(\rho)$  are of the form  $\rho = \chi_F \otimes \rho_0 = (X, (\chi_F \circ N_{K/F})\chi_0)$ , where  $\chi_F : F^\times \rightarrow \mathbb{C}^\times$ . Then we have (cf. [5], p. 305, equation (5))

$$(3.32) \quad \text{sw}_F(\rho) = \text{sw}_F(\chi_F \otimes \rho_0) = \sqrt{[F^\times : \text{Rad}(X)]} \cdot \max\{j(\chi_F), j(X)\}.$$

For minimal conductor U-isotopic Heisenberg representation we have the following proposition.

**Proposition 3.21.** *Let  $\rho = \rho(X_\eta, \chi_K)$  be a  $U$ -isotropic Heisenberg representation of  $G_F$  of minimal conductor. Then we have the following conductor relation*

$$\begin{aligned} j(X_\eta) &= j(\eta), \text{ sw}_F(\rho) = \dim(\rho) \cdot j(X_\eta) = \#\eta \cdot j(\eta), \\ a_F(\rho) &= \text{sw}_F(\rho) + \dim(\rho) = \#\eta(j(\eta) + 1) = \#\eta \cdot a_F(\eta). \end{aligned}$$

*Proof.* From [8], on p. 126, Proposition 4(i) and Proposition 5(ii), and  $U \wedge U = U^1 \wedge U^1$ , we see the injection  $U^i \wedge F^\times \subseteq UU^i$  induces a natural isomorphism

$$U^i \wedge \langle \pi_F \rangle \cong UU^i / UU^i \cap (U \wedge U)$$

for all  $i \geq 0$ .

Now let  $j(X_\eta) = n - 1$ , hence  $X_\eta|_{UU^n} = 1$  but  $X_\eta|_{UU^{n-1}} \neq 1$ . This gives  $X_\eta|_{U^n \wedge \langle \pi_F \rangle} = 1$  but  $X_\eta|_{U^{n-1} \wedge \langle \pi_F \rangle} \neq 1$ . Now from equation (3.19) we can conclude that  $\eta(x) = 1$  for all  $x \in U^n$  but  $\eta(x) \neq 1$  for  $x \in U^{n-1}$ . Hence

$$j(\eta) = n - 1 = j(X_\eta).$$

Again from the definition of  $j(\chi)$ , where  $\chi$  is a character of  $F^\times$ , we can see that  $j(\chi) = a(\chi) - 1$ , i.e.,  $a(\chi) = j(\chi) + 1$ .

From equation (3.31) we obtain:

$$\text{sw}_F(\rho) = \dim(\rho) \cdot j(X_\eta) = \#\eta \cdot j(\eta),$$

since  $\dim(\rho) = \#\eta$  and  $j(X_\eta) = j(\eta)$ . Finally, from equation (3.27) for  $\rho$  (here  $< 1, \rho >_{G_0} = 0$ ), we have

$$(3.33) \quad a_F(\rho) = \text{sw}_F(\rho) + \dim(\rho) = \#\eta \cdot j(\eta) + \#\eta = \#\eta \cdot (j(\eta) + 1) = \#\eta \cdot a_F(\eta).$$

□

By using the equation (3.28) in our Heisenberg setting, we have the following proposition.

**Proposition 3.22.** *Let  $\rho = \rho(Z, \chi_\rho) = \rho(X, \chi_K)$  be a Heisenberg representation of the absolute Galois group  $G_F$  of a field  $F/\mathbb{Q}_p$  of dimension  $m$ . Let  $E/F$  be any subextension in  $K/F$  corresponding to a maximal isotropic subgroup for  $X$ . Then*

$$a_F(\rho) = a_F(\text{Ind}_{E/F}(\chi_E)), \quad m \cdot a_F(\rho) = a_F(\text{Ind}_{K/F}(\chi_K)).$$

As a consequence we have

$$a(\chi_K) = e_{K/E} \cdot a(\chi_E) - d_{K/E}.$$

*Proof.* We know that  $\rho = \text{Ind}_{E/F}(\chi_E)$  and  $m \cdot \rho = \text{Ind}_{K/F}(\chi_K)$ . By the definition of Artin conductor we can write

$$a_F(\dim(\rho) \cdot \rho) = \dim(\rho) \cdot a_F(\rho) = m \cdot a_F(\text{Ind}_{E/F}(\chi_E)).$$

Since  $K/E/F$  is a tower of Galois extensions with  $[K : E] = m = e_{K/E} f_{K/E}$ , we have the transitivity relation of different (cf. [15], p. 51, Proposition 8)

$$\mathcal{D}_{K/F} = \mathcal{D}_{K/E} \cdot \mathcal{D}_{E/F}.$$

Now from the definition of different of a Galois extension, and taking  $K$ -valuation we obtain:

$$(3.34) \quad d_{K/F} = d_{K/E} + e_{K/E} \cdot d_{E/F}.$$



Now by using equation (3.28) we have:

$$(3.35) \quad m \cdot a_F(\text{Ind}_{E/F}(\chi_E)) = m \cdot f_{E/F} (d_{E/F} + a(\chi_E)) = m \cdot f_{E/F} \cdot d_{E/F} + e_{K/E} \cdot f_{K/F} \cdot a(\chi_E),$$

and

$$(3.36) \quad a_F(\text{Ind}_{K/F}(\chi_K)) = f_{K/F} \cdot (d_{K/F} + a(\chi_K)) = f_{K/F} \cdot d_{K/F} + f_{K/F} \cdot a(\chi_K).$$

By using equation (3.34), from equations (3.35), (3.36), we have

$$a(\chi_K) = e_{K/E} \cdot a(\chi_E) - d_{K/E}$$

□

Now by combining Proposition 3.22 with Proposition 3.21, we get the following result.

**Lemma 3.23.** *Let  $\rho = \rho(X_\eta, \chi_K)$  be a  $U$ -isotopic Heisenberg representation of the absolute Galois group  $G_F$  of a non-archimedean local field  $F$ . Let  $K = K_\eta$  correspond to the radical of  $X_\eta$ , and let  $E_1/F$  be the maximal unramified subextension, and  $E/F$  be any maximal cyclic and totally ramified subextension in  $K/F$ . Let  $m$  denote the order of  $\eta$ . Then  $\rho$  is induced by  $\chi_{E_1}$  or by  $\chi_E$  respectively, and we have*

- (1)  $a_E(\chi_E) = m \cdot a(\eta) - d_{E/F}$ ,
- (2)  $a_{E_1}(\chi_{E_1}) = a(\eta)$ ,
- (3) and for the character  $\chi_K \in \widehat{K^\times}$ ,

$$a_K(\chi_K) = m \cdot a(\eta) - d_{K/F}.$$

Moreover,  $a_E(\chi_E) = a_K(\chi_K)$ .

*Proof.* Proof of these assertions follows from equation (3.28) and Proposition 3.21. When  $\rho = \text{Ind}_{E/F}(\chi_E)$ , where  $E/F$  is a maximal cyclic and totally ramified subextension in  $K/F$ , from equation (3.28) we have

$$\begin{aligned} a_F(\rho) &= m \cdot a(\eta) \quad \text{using Proposition 3.21,} \\ &= f_{E/F} \cdot (d_{E/F} \cdot 1 + a_E(\chi_E)), \quad \text{since } \rho = \text{Ind}_{E/F}(\chi_E) \\ &= 1 \cdot (d_{E/F} + a_E(\chi_E)). \end{aligned}$$

because  $E/F$  is totally ramified, hence  $f_{E/F} = 1$ . This implies  $a_E(\chi_E) = m \cdot a(\eta) - d_{E/F}$ .

Similarly, when  $\rho = \text{Ind}_{E_1/F}(\chi_{E_1})$ , where  $E_1/F$  is the maximal unramified subextension in  $K/F$ , hence  $f_{E_1/F} = m$  and  $d_{E_1/F} = 0$ , by using equation (3.28) we obtain  $a_{E_1}(\chi_{E_1}) = a(\eta)$ .

Again from Proposition 3.22 we have

$$a_K(\chi_K) = m \cdot a(\chi_{E_1}) - d_{K/E_1} = m \cdot a(\eta) - d_{K/F}.$$

Finally, since  $E/F$  is a maximal cyclic totally ramified implies  $K/E$  is unramified and therefore

$$d_{E/F} = d_{K/F}, \quad \text{and hence } a_E(\chi_E) = a_K(\chi_K).$$

□

*Remark 3.24.* Assume that we are in the dimension  $m = \# \eta$  prime to  $p$  case. Then from Corollary 3.12,  $\eta$  must be a character of  $U/U^1$  (for  $U = U_F$ ), hence

$$a(\eta) = 1 \quad a_F(\rho_0) = m.$$

Therefore in this case the minimal conductor of  $\rho$  is  $m$ , hence it is equal to the dimension of  $\rho$ .

From the above Lemma 3.23, in this case we have

$$a_{E_1}(\chi_{E_1}) = a(\eta) = 1.$$

And  $K/F, E/F$  are tamely ramified of ramification exponent  $e_{K/F} = m$ , hence

$$a_E(\chi_E) = a_K(\chi_K) = m \cdot a(\eta) - d_{K/F} = m - (e_{K/F} - 1) = m - (m - 1) = 1.$$

Thus we can conclude that in this case all three characters (i.e.,  $\chi_{E_1}, \chi_E$ , and  $\chi_K$ ) are of conductor 1.

In the general case  $a_{E_1}(\chi_{E_1}) = a(\eta)$  and

$$a_E(\chi_E) = a_K(\chi_K) = m \cdot a(\eta) - d,$$

where  $d = d_{E/F} = d_{K/F}$ , conductors will be different.

In general, if  $\rho = \rho_0 \otimes \chi_F$ , where  $\rho_0$  is a finite dimensional minimal conductor representation of  $G_F$ , and  $\chi_F \in \widehat{F^\times}$ , then we have the following result.

**Lemma 3.25.** *Let  $\rho_0$  be a finite dimensional representation of  $G_F$  of minimal conductor. Then we have*

$$(3.37) \quad a_F(\rho) = \dim(\rho_0) \cdot a_F(\chi_F),$$

where  $\rho = \rho_0 \otimes \chi_F = \rho(X_\eta, (\chi_F \circ N_{K/F})\chi_0)$  and  $\chi_F \in \widehat{F^\times}$  with  $a(\chi_F) > \frac{a(\rho_0)}{\dim(\rho)}$ .

*Proof.* From equation (3.29) we have  $a_F(\rho_0) = \dim(\rho_0) \cdot (1 + j(\rho_0))$ . By the given condition  $\rho_0$  is of minimal conductor. So for representation  $\rho = \rho_0 \otimes \chi_F$ , we have

$$\begin{aligned} a_F(\rho) &= a_F(\rho_0 \otimes \chi_F) = \dim(\rho_0) \cdot (1 + \max\{j(\rho_0), j(\chi_F)\}) \\ &= \dim(\rho_0) \cdot \max\{1 + j(\chi_F), 1 + j(\rho_0)\} \\ &= \dim(\rho_0) \cdot \max\{a(\chi_F), 1 + j(\rho_0)\} \\ &= \dim(\rho_0) \cdot a_F(\chi_F), \end{aligned}$$

because by the given condition

$$a(\chi_F) > \frac{a(\rho_0)}{\dim(\rho_0)} = \frac{\dim(\rho_0) \cdot (1 + j(\rho_0))}{\dim(\rho_0)} = 1 + j(\rho_0).$$

□

**Proposition 3.26.** *Let  $\rho = \rho(X, \chi_K)$  be a Heisenberg representation dimension  $m$  of the absolute Galois group  $G_F$  of a non-archimedean local field  $F$ . Then  $m|a_F(\rho)$  if and only if:  $X$  is  $U$ -isotropic, or (if  $X$  is not  $U$ -isotropic)  $a_F(\rho)$  is with respect to  $X$  not the minimal conductor.*

*Proof.* From the above Lemma 3.25 we know that if  $\rho$  is not minimal, then  $a_F(\rho)$  is always a multiple of the dimension  $m$ . So now we just have to check for minimal conductors. In the U-isotropic case the minimal conductor is multiple of the dimension (cf. Proposition 3.21).

Finally, suppose that  $X$  is not U-isotropic, i.e.,  $X|_{U \wedge U} = X|_{U^1 \wedge U^1} \neq 1$ , because  $U \wedge U = U^1 \wedge U^1$  (see the Remark on p. 126 of [8]). We also know that  $UU^i = (UU^i \cap U^1 \wedge U^1) \times (U^i \wedge < \pi_F >)$  (cf. [8], p. 126, Proposition 5(ii)). In Proposition 5 of [8], we observe that all the jumps  $v$  in the filtration  $\{UU^i \cap (U^1 \wedge U^1)\}, i \in \mathbb{R}_+$  are not **integers with**  $v > 1$ . This shows that  $j(X)$  is also not an integer, hence  $a_F(\rho_0)$  is not multiple of the dimension. This implies the conductor  $a_F(\rho)$  is not minimal.  $\square$

Let  $\rho = \rho(X, \chi_K)$  be a Heisenberg representation of the absolute Galois group  $G_F$ . Then from equation (3.6), we have

$$\dim(\rho) = \sqrt{[K : F]} = \sqrt{[F^\times : \mathcal{N}_{K/F}]},$$

when  $\mathcal{N}_{K/F} = \text{Rad}(X)$ .

**Lemma 3.27.** *Let  $\rho = (Z_\rho, \chi) = \rho(X_\rho, \chi)$  be a Heisenberg representation of the absolute Galois group  $G_F$  of a non-archimedean local field  $F/\mathbb{Q}_p$ . Then following are equivalent:*

- (1)  $\dim(\rho)$  is prime to  $p$ .
- (2)  $\dim(\rho)$  is a divisor of  $q_F - 1$ .
- (3) The alternating character  $X_\rho$  is U-isotropic and  $X_\rho = X_\eta$  for a character  $\eta$  of  $U_F/U_F^1$ .

*Proof.* From Corollary 3.12 we know that all Heisenberg representations of dimensions prime to  $p$ , are U-isotropic representations of the form  $\rho = \rho(X_\eta, \chi)$ , where  $\eta : U_F/U_F^1 \rightarrow \mathbb{C}^\times$ , and the dimensions  $\dim(\rho) = \#\eta$ .

Thus if  $\dim(\rho)$  is prime to  $p$ , then  $\dim(\rho) = \#\eta$  is a divisor of  $q_F - 1$ . And if  $\dim(\rho)$  is a divisor of  $q_F - 1$ , then  $\gcd(p, \dim(\rho)) = 1$ . Then from Corollary 3.12, the alternating character  $X_\rho$  is U-isotropic and  $X_\rho = X_\eta$  for a character  $\eta \in \widehat{U_F/U_F^1}$ .

Finally, if  $\rho = \rho(X_\rho, \chi_K) = \rho(X_\rho, \chi_K)$  be a Heisenberg representation of  $G_F$  for a character  $\eta$  of  $U_F/U_F^1$ , then from Corollary 3.12, we know that  $\dim(\rho)$  is prime to  $p$ .  $\square$

For giving invariant formula of  $W(\rho)$ , we need to know the explicit dimension formula of  $\rho$ . In the following theorem we give the general dimension formula of a Heisenberg representation.

**Theorem 3.28 (Dimension).** *Let  $F/\mathbb{Q}_p$  be a local field and  $G_F$  be the absolute Galois group of  $F$ . If  $\rho$  is a Heisenberg representation of  $G_F$ , then  $\dim(\rho) = p^n \cdot d'$ , where  $n \geq 0$  is an integer and where the prime to  $p$  factor  $d'$  must divide  $q_F - 1$ .*

*Proof.* By the definition of Heisenberg representation  $\rho$  we have the relation

$$[[G_F, G_F], G_F] \subseteq \text{Ker}(\rho).$$

Then we can consider  $\rho$  as a representation of  $G := G_F/[[G_F, G_F], G_F]$ . Since  $[x, g] \in [[G_F, G_F], G_F]$  for all  $x \in [G_F, G_F]$  and  $g \in G_F$ , we have  $[G, G] = [G_F, G_F]/[[G_F, G_F], G_F] \subseteq Z(G)$ , hence  $G$  is a two-step nilpotent group.

We know that each nilpotent group is isomorphic to the direct product of its Sylow subgroups. Therefore we can write

$$G = G_p \times G_{p'},$$

where  $G_p$  is the Sylow  $p$ -subgroup, and  $G_{p'}$  is the direct product of all other Sylow subgroups. Therefore each irreducible representation  $\rho$  has the form  $\rho = \rho_p \otimes \rho_{p'}$ , where  $\rho_p$  and  $\rho_{p'}$  are irreducible representations of  $G_p$  and  $G_{p'}$  respectively.

We also know that finite  $p$ -groups are nilpotent groups, and direct product of a finite number of nilpotent groups is again a nilpotent group. So  $G_p$  and  $G_{p'}$  are both two-step nilpotent group because  $G$  is a two-step nilpotent group. Therefore the representations  $\rho_p$  and  $\rho_{p'}$  are both Heisenberg representations of  $G_p$  and  $G_{p'}$  respectively.

Now to prove our assertion, we have to show that  $\dim(\rho_p)$  can be an arbitrary power of  $p$ , whereas  $\dim(\rho_{p'})$  must divide  $q_F - 1$ . Since  $\rho_p$  is an **irreducible** representation of  $p$ -group  $G_p$ , so the dimension of  $\rho_p$  is some  $p$ -power.

Again from the construction of  $\rho_{p'}$  we can say that  $\dim(\rho_{p'})$  is **prime** to  $p$ . Then from Lemma 3.27  $\dim(\rho_{p'})$  is a divisor of  $q_F - 1$ .

This completes the proof. □

*Remark 3.29. (1).* Let  $V_F$  be the wild ramification subgroup of  $G_F$ . We can show that  $\rho|_{V_F}$  is irreducible if and only if  $Z_\rho = G_K \subset G_F$  corresponds to an abelian extension  $K/F$  which is totally ramified and wildly ramified<sup>4</sup> (cf. [5], p. 305). If  $N := N_{K/F}(K^\times)$  is the subgroup of norms, then this means that  $N \cdot U_F^1 = F^\times$ , in other words,

$$F^\times / N = N \cdot U_F^1 / N = U_F^1 / N \cap U_F^1,$$

where  $N$  can be also considered as the radical of  $X_\rho$ . So we can consider the alternating character  $X_\rho$  on the principal units  $U_F^1 \subset F^\times$ . Then

$$\dim(\rho) = \sqrt{[F^\times : N]} = \sqrt{[U_F^1 : N \cap U_F^1]},$$

is a power of  $p$ , because  $U_F^1$  is a pro- $p$ -group.

Here we observe: If  $\rho = \rho(X, \chi_K)$  with  $\rho|_{V_F}$  stays irreducible, then  $\dim(\rho) = p^n$ ,  $n \geq 1$  and  $K/F$  is a totally and **wildly** ramified. But there is a **big** class of Heisenberg representations  $\rho$  such that  $\dim(\rho) = p^n$  is a  $p$ -power, but which are not wild representations (see the Definition 3.8 of U-isotropic).

**(2).** Let  $\rho = \rho(X, \chi_K)$  be a Heisenberg representation of the absolute Galois group  $G_F$  of dimension  $d > 1$  which is prime to  $p$ . Then from above Lemma 3.27, we have  $d | q_F - 1$ . For this representation  $\rho$ , here  $K/F$  must be tame if  $\text{Rad}(X) = \mathcal{N}_{K/F}$  (cf. [12], p. 115).

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<sup>4</sup>Group theoretically, if  $\rho|_{V_F} = \text{Ind}_H^{G_F}(\chi_H)|_{V_F}$  is irreducible, then from Section 7.4 of [16], we can say  $G_F = H \cdot V_F$ . Here  $H = G_L$ , where  $L$  is a certain extension of  $F$ , and  $V_F = G_{F_{mt}}$  where  $F_{mt}/F$  is the maximal tame extension of  $F$ . Therefore  $G_F = H \cdot V_F$  is equivalent to  $F = L \cap F_{mt}$  that means the extension  $L/F$  must be totally ramified and wildly ramified, and  $[G_F : H] = [L : F] = |V_F|$ . We know that the wild ramification subgroup  $V_F$  is a pro- $p$ -group (cf. [12], p. 106). Then  $\dim(\rho)$  is a power of  $p$ .

4. Invariant formula for  $W(\rho)$ 

**Lemma 4.1.** *Let  $\rho = \rho(Z, \chi_Z)$  be a Heisenberg representation of the local Galois group  $G = \text{Gal}(L/F)$  of odd dimension. Let  $H$  be a maximal isotropic subgroup for  $\rho$  and  $\chi_H \in \widehat{H}$  with  $\chi_H|_Z = \chi_Z$  then:*

$$(4.1) \quad W(\rho) = W(\chi_H), \quad W(\rho)^{\dim(\rho)} = W(\chi_Z),$$

and

$$(4.2) \quad W(\chi_H)^{[H:Z]} = W(\chi_Z).$$

*Proof.* From the construction of Heisenberg representation  $\rho$  of  $G$  we have

$$\rho = \text{Ind}_H^G(\chi_H), \quad \dim(\rho) \cdot \rho = \text{Ind}_Z^G(\chi_Z).$$

This implies that  $W(\rho) = \lambda_H^G \cdot W(\chi_H)$  and  $W(\rho)^{\dim(\rho)} = \lambda_Z^G \cdot W(\chi_Z)$ .

Since  $\dim(\rho)$  is odd we may apply now Lemma 3.4 on p. 10 of [25], and we obtain

$$\lambda_H^G = \lambda_Z^G = 1.$$

So, we have  $W(\rho) = \lambda_H^G(W) \cdot W(\chi_H) = W(\chi_H)$ . Similarly, we have  $W(\rho)^{\dim(\rho)} = W(\chi_Z)$ .

Moreover, it is easy to see<sup>5</sup> that  $W(\text{Ind}_Z^H(\chi_Z)) = W(\chi_H)^{[H:Z]}$ . By the given condition,  $[H : Z] = \dim(\rho)$  is odd, hence  $\lambda_Z^H = 1$ , then we have

$$(4.3) \quad W(\chi_H)^{[H:Z]} = W(\text{Ind}_Z^H(\chi_Z)) = W(\chi_Z).$$

□

*Remark 4.2.* Related to  $G \supset H \supset Z$  we have the base fields  $F \subset E \subset K$ , and  $\chi_Z$  is the restriction of  $\chi_H$ . In arithmetic terms this means:

$$\chi_K = \chi_E \circ N_{K/E}.$$

So in arithmetic terms of  $W(\text{Ind}_Z^G(\chi_Z)) = W(\text{Ind}_H^G(\chi_H))^{[G:H]}$  is as follows:

$$W(\text{Ind}_{K/F}(\chi_K), \psi) = W(\text{Ind}_{E/F}(\chi_E), \psi)^{[K:E]}.$$

Then we can conclude that

$$\lambda_{K/E} \cdot W(\chi_K, \psi_K) = W(\chi_E, \psi_E)^{[K:F]}.$$

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<sup>5</sup>We have

$$d \cdot \rho = \text{Ind}_Z^G \chi_Z = \text{Ind}_H^G(\text{Ind}_Z^H \chi_Z),$$

and  $\text{Ind}_Z^H \chi_Z$  of dimension  $d = [H : Z]$ . Therefore:

$$W(\rho)^d = (\lambda_H^G)^d \cdot W(\text{Ind}_Z^H \chi_Z).$$

On the other hand  $W(\rho) = \lambda_H^G \cdot W(\chi_H)$  implies

$$W(\rho)^d = (\lambda_H^G)^d \cdot W(\chi_H)^d.$$

Now comparing these two expressions for  $W(\rho)^d$  we see that

$$W(\chi_H)^d = W(\text{Ind}_Z^H \chi_Z).$$

If the dimension  $\dim(\rho) = [K : E]$  is odd, we have  $\lambda_{K/E} = 1$ , because  $K/E$  is Galois. Then we obtain

$$(4.4) \quad W(\chi_E, \psi_E)^{[K:E]} = W(\chi_E \circ N_{K/E}, \psi_E \circ \text{Tr}_{K/E}).$$

The formula (4.4) is known as a **Davenport-Hasse** relation (cf. [23], p. 197, Theorem 5.14).

**Corollary 4.3.** *Let  $\rho = \rho(Z, \chi_Z)$  be a Heisenberg representation of a local Galois group  $G$ . Let  $\dim(\rho) = d$  be odd. Let the order of  $W(\chi_Z)$  be  $n$  (i.e.,  $W(\chi_Z)^n = 1$ ). If  $d$  is prime to  $n$ , then  $d^{\varphi(n)} \equiv 1 \pmod{n}$ , and*

$$W(\rho) = W(\chi_Z)^{\frac{1}{d}} = W(\chi_Z)^{d^{\varphi(n)-1}},$$

where  $\varphi(n)$  is the Euler's totient function of  $n$ .

*Proof.* By our assumption, here  $d$  and  $n$  are coprime. Therefore from **Euler's theorem** we can write

$$d^{\varphi(n)} \equiv 1 \pmod{n}.$$

This implies  $d^{\varphi(n)} - 1$  is a multiple of  $n$ .

Here  $d$  is odd, then from the above Lemma 4.1 we have  $W(\rho)^d = W(\chi_Z)$ . So we obtain

$$W(\rho) = W(\chi_Z)^{\frac{1}{d}} = W(\chi_Z)^{d^{\varphi(n)-1}},$$

since  $d^{\varphi(n)} - 1$  is a multiple of  $n$ , and by assumption  $W(\chi_Z)^n = 1$ .  $\square$

We observe that when  $\dim(\rho) = d$  is odd, if we take second part of the equation (4.1), we have  $W(\rho) = W(\chi_Z)^{\frac{1}{d}}$ , but it is not well-defined in general. Here we have to make precise which root  $W(\chi_Z)^{\frac{1}{d}}$  really occurs. That is why, giving invariant formula of  $W(\rho)$  using  $\lambda$ -functions computation is difficult. In the following theorem we give an invariant formula of local constant for Heisenberg representation.

**Theorem 4.4.** *Let  $\rho = \rho(X, \chi_K)$  be a Heisenberg representation of the absolute Galois group  $G_F$  of a local field  $F/\mathbb{Q}_p$  of dimension  $d$ . Let  $\psi_F$  be the canonical additive character of  $F$  and  $\psi_K := \psi_F \circ \text{Tr}_{K/F}$ . Denote  $\mu_{p^\infty}$  as the group of roots of unity of  $p$ -power order and  $\mu_n$  as the group of  $n$ -th roots of unity, where  $n > 1$  is an integer.*

(1) *When the dimension  $d$  is odd, we have*

$$W(\rho) \equiv W(\chi_\rho)' \pmod{\mu_d},$$

where  $W(\chi_\rho)'$  is any  $d$ -th root of  $W(\chi_K, \psi_K)$ .

(2) *When the dimension  $d$  is even, we have*

$$W(\rho) \equiv W(\chi_\rho)' \pmod{\mu_{d'}},$$

where  $d' = \text{lcm}(4, d)$ .

*Proof. (1).* We know that the lambda functions are always fourth roots of unity. In particular, when degree of the Galois extension  $K/F$  is odd, from Theorem 2.3 we have  $\lambda_{K/F} = 1$ . For proving our assertions we will use these facts about  $\lambda$ -functions.

We know that  $\dim(\rho) \cdot \rho = \text{Ind}_{K/F}(\chi_K)$ , where by class field theory  $\chi_K \leftrightarrow \chi_\rho$  is a character of  $K^\times$ . When  $d$  is odd, we can write

$$W(\rho)^d = \lambda_{K/F} \cdot W(\chi_K, \psi_K) = W(\chi_K, \psi_K).$$

Now let  $W(\chi_\rho)'$  be any  $d$ -th root of  $W(\chi_K, \psi_K)$ . Then we have

$$W(\rho)^d = W(\chi_\rho)'^d,$$

hence  $\frac{W(\rho)}{W(\chi_\rho)'}$  is a  $d$ -th root of unity. Therefore we have

$$W(\rho) \equiv W(\chi_\rho)' \pmod{\mu_d}.$$

(2). Similarly, we can give invariant formula for even degree Heisenberg representations. When the dimension  $d$  of  $\rho$  is even, we have

$$(4.5) \quad W(\rho)^d = \lambda_{K/F} \cdot W(\chi_K, \psi_K) \equiv W(\chi_K, \psi_K) \pmod{\mu_4},$$

because  $\lambda_{K/F}$  is a fourth root of unity. Now let  $W(\chi_\rho)'$  be any  $d$ -th root of  $W(\chi_K, \psi_K)$ , hence  $W(\chi_K, \psi_K) = W(\chi_\rho)'^d$ . Then from equation (4.5) we have

$$\left( \frac{W(\rho)}{W(\chi_\rho)'} \right)^d \equiv 1 \pmod{\mu_4}.$$

Therefore we can conclude that

$$(4.6) \quad W(\rho) \equiv W(\chi_\rho)' \pmod{\mu_{d'}},$$

where  $d' = \text{lcm}(4, d)$ .

□

When dimension of a Heisenberg representation  $\rho = \rho(X, \chi_K)$  of  $G_F$  is prime to  $p$ , then from Lemma 3.27 we can say that  $X = X_\eta$  is U-isotropic with  $\eta : U_F/U_F^1 \rightarrow \mathbb{C}^\times$ . Again from Remark 3.24 we observe that  $a(\chi_K) = 1$  when  $\rho$  is of minimal conductor. In the following lemma for minimal conductor  $\rho$  with dimension prime to  $p$ , we show that  $W(\rho)$  is a root of unity.

**Lemma 4.5.** *Let  $\rho = \rho(X, \chi_K)$  be a minimal conductor Heisenberg representation with respect to  $X$  of the absolute Galois group  $G_F$  of a non-archimedean local field  $F/\mathbb{Q}_p$ . If dimension  $\dim(\rho)$  is prime to  $p$ , then  $W(\rho)$  is always a root of unity.*

*Proof.* Assume that  $\dim(\rho) = d$  and  $\gcd(d, p) = 1$ . Then from Lemma 3.27, we can say that  $\rho = \rho(X, \chi_K) = \rho(X_\eta, \chi_K)$  is a U-isotropic with  $a(\eta) = 1$ . Since  $\rho$  is of minimal conductor, from Remark 3.24 we have  $a(\chi_K) = 1$ .

From equation (3.5) we also know that:

$$d \cdot \rho = \text{Ind}_{K/F}(\chi_K).$$

Then we can write

$$\begin{aligned} W(\rho)^d &= \lambda_{K/F} \cdot W(\chi_K) \\ &= \lambda_{K/F} \cdot q_K^{-\frac{1}{2}} \sum_{x \in U_K/U_K^1} \chi_K^{-1}(x/c) \psi_K(x/c) \\ (4.7) \quad &= \lambda_{K/F} \cdot q_K^{-\frac{1}{2}} \tau(\chi_K), \end{aligned}$$

where  $c = \pi_K^{1+n(\psi_K)}$ ,  $\psi_K = \psi_F \circ \text{Tr}_{K/F}$ , the canonical character of  $K$  and

$$(4.8) \quad \tau(\chi_K) = \sum_{x \in U_K/U_K^1} \chi_K^{-1}(x/c) \psi_K(x/c).$$

Since  $U_K/U_K^1 \cong k_K^\times$ ,  $a(\chi_K) = 1$ , and  $n(\frac{1}{c} \cdot \psi_K) = -1$ , we can consider  $\tau(\chi_K)$  as a classical Gauss sum of  $\chi_K$ . We also know that  $|\tau(\chi_K)| = q_K^{\frac{1}{2}}$  (cf. [14], p. 30, Proposition 2.2(i)).

Moreover, here we have  $f_{K/F} = e_{K/F} = d$ , hence  $f_{K/\mathbb{Q}_p} \geq d$ . So here we have  $q_K = p^{f_{K/\mathbb{Q}_p}} \geq p^d$ . Then from Theorem 1.6.2 on p. 33 of [1], we can write

$$\tau(\chi_K) = q_K^{\frac{1}{2}} \cdot \gamma,$$

where  $\gamma$  is a certain root of unity.

We also know that  $\lambda_{K/F}^4 = 1$ , then from the equation (4.7) we obtain:

$$(4.9) \quad W(\rho)^{4dn} = \gamma^{4n} = 1,$$

where  $n$  is the order of  $\gamma$ .

This completes the proof. □

*Remark 4.6.* As to the computation of  $W(\rho) = W(\rho(X, \chi_K))$  we also can precisely say what an unramified twist will do by the formula of local constant of unramified character twist (cf. [20], p. 15, (3.4.5)). Let  $\omega_{K,s}$  be an unramified character of  $K^\times$  such that  $\omega_{K,s}|_{F^\times} = \omega_{F,s}$ , then we have

$$(4.10) \quad \omega_{F,s} \otimes \rho(X, \chi_K) = \rho(X, \omega_{K,s} \cdot \chi_K), \quad W(\rho(X, \omega_{K,s} \cdot \chi_K)) = \omega_{F,s}(c_{\rho,\psi}) \cdot W(\rho(X, \chi_K)).$$

Therefore the question: Is  $W(\rho)$  a root of unity or not?, is completely under control if we do unramified twists. In particular, unramified twists of finite order cannot change the answer.

In the following theorem we give an invariant formula for  $W(\rho, \psi)$ , where  $\rho = \rho(X, \chi_K)$  is a minimal conductor Heisenberg representation of the absolute Galois group  $G_F$  of a local field  $F/\mathbb{Q}_p$  of dimension  $m$  which is prime to  $p$ .

**Theorem 4.7.** *Let  $\rho = \rho(X, \chi_K)$  be a minimal conductor Heisenberg representation of the absolute Galois group  $G_F$  of a non-archimedean local field  $F/\mathbb{Q}_p$  of dimension  $m$  with  $\gcd(m, p) = 1$ . Let  $\psi$  be a nontrivial additive character of  $F$ . Then*

$$(4.11) \quad W(\rho, \psi) = R(\psi, c) \cdot L(\psi, c),$$

where

$$R(\psi, c) := \lambda_{E/F}(\psi) \Delta_{E/F}(c),$$

is a fourth root of unity that depends on  $c \in F^\times$  with  $\nu_F(c) = 1 + n(\psi)$  but not on the totally ramified cyclic subextension  $E/F$  in  $K/F$ , and

$$L(\psi, c) := \det(\rho)(c) q_F^{-\frac{1}{2}} \sum_{x \in k_F^\times} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(x) \cdot (c^{-1}\psi)(mx),$$

where  $E_1/F$  is the unramified extension of  $F$  of degree  $m$ .

Before proving this Theorem 4.7 we need to prove the following lemma.



**Lemma 4.8.** *(With the notation of the above theorem)*

(1) *Let  $E/F$  be any totally ramified cyclic extension of degree  $m$  inside  $K/F$ . Then:*

$$\Delta_{E/F}(\epsilon) =: \Delta(\epsilon), \quad \text{for all } \epsilon \in U_F,$$

*does not depend on  $E$  if we restrict to units of  $F$ .*

(2) *We have  $L(\psi, \epsilon c) = \Delta(\epsilon)L(\psi, c)$ , and therefore changing  $c$  by unit we see that*

$$\Delta_{E/F}(\epsilon c) \cdot L(\psi, \epsilon c) = \Delta(\epsilon)^2 \Delta_{E/F}(c) \cdot L(\psi, c) = \Delta_{E/F}(c)L(\psi, c).$$

(3) *We also have the transformation rule  $R(\psi, \epsilon c) = \Delta(\epsilon) \cdot R(\psi, c)$ .*

*Proof. (1).* Denote  $G := \text{Gal}(E/F)$ . By class field theory we know that

$$(4.12) \quad \Delta_{E/F} = \begin{cases} \omega_{E'/F} & \text{when } \text{rk}_2(G) = 1 \\ 1 & \text{when } \text{rk}_2(G) = 0, \end{cases}$$

where  $E'/F$  is a uniquely determined quadratic extension inside  $E/F$ , and  $\omega_{E'/F}$  is the quadratic character of  $F^\times$  which corresponds to the extension  $E'/F$  by class field theory.

When  $m$  is odd, i.e.,  $\text{rk}_2(G) = 0$ , hence  $\Delta_{E/F} \cong 1$ . So for odd case, the assertion (1) is obvious.

When  $m$  is even, we choose two different totally ramified cyclic subextensions, namely  $L_1/F$ ,  $L_2/F$ , in  $K/F$  of degree  $m$ . Then we can write for all  $\epsilon \in U_F$ ,

$$\Delta_{L_1/F}(\epsilon) = \omega_{E'/F}(\epsilon) = \eta(\epsilon) \cdot \omega_{E'/F}(\epsilon) = \omega_{E'/F}(\epsilon) = \Delta_{L_2/F}(\epsilon),$$

where  $\eta$  is the unramified quadratic character of  $F^\times$ . This proves that  $\Delta_{E/F}$  does not depend on  $E$  if we restrict to  $U_F$ .

(2). From Proposition 3.7 we know that  $\det(\rho)(x) = \Delta_{E/F}(x) \cdot \chi_K \circ N_{K/E}^{-1}(x)$  for all  $x \in F^\times$ . Let  $E_1/F$  be the unramified subextension in  $K/F$  of degree  $m$ . Then we have  $EE_1 = K$  and

$$N_{K/E}|_{E_1} = N_{E_1/F}, \quad (E_1^\times)_F \subseteq K_E^\times \subset \text{Ker}(\chi_K).$$

Moreover  $U_F \subset \mathcal{N}_{E_1/F}$  and therefore we may write  $N_{K/E}^{-1}(\epsilon) = N_{E_1/F}^{-1}(\epsilon)$  for all  $\epsilon \in U_F$ . Now we can write:

$$\begin{aligned} L(\psi, \epsilon c) &= \det(\rho)(\epsilon c) q_F^{-\frac{1}{2}} \sum_{x \in k_F^\times} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(x) \cdot (c^{-1}\psi)(mx/\epsilon) \\ &= \Delta_{E/F}(\epsilon) \chi_K \circ N_{K/E}^{-1}(\epsilon) \det(\rho)(c) q_F^{-\frac{1}{2}} \sum_{x \in k_F^\times} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(\epsilon x) \cdot (c^{-1}\psi)(mx) \\ &= \Delta(\epsilon) \chi_K \circ N_{E_1/F}^{-1}(\epsilon \epsilon^{-1}) \det(\rho)(c) q_F^{-\frac{1}{2}} \sum_{x \in k_F^\times} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(x) \cdot (c^{-1}\psi)(mx) \\ &= \Delta(\epsilon) \cdot \det(\rho)(c) q_F^{-\frac{1}{2}} \sum_{x \in k_F^\times} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(x) \cdot (c^{-1}\psi)(mx) \\ &= \Delta(\epsilon) \cdot L(\psi, c). \end{aligned}$$

This implies that

$$\Delta_{E/F}(\epsilon c) \cdot L(\psi, \epsilon c) = \Delta(\epsilon)^2 \Delta_{E/F}(c) \cdot L(\psi, c) = \Delta_{E/F}(c)L(\psi, c).$$

(3). By the definition of  $R(\psi, c)$  we can write:

$$\begin{aligned} R(\psi, \epsilon c) &= \lambda_{E/F}(\psi) \Delta_{E/F}(\epsilon c) = \lambda_{E/F}(\psi) \Delta_{E/F}(\epsilon) \Delta_{E/F}(c) \\ &= \Delta(\epsilon) \lambda_{E/F}(\psi) \Delta_{E/F}(c) = \Delta(\epsilon) \cdot R(\psi, c). \end{aligned}$$

□

Now we are in a position to give a proof of Theorem 4.7 by using Lemma 4.8.

**Proof of Theorem 4.7.** By the given conditions:  $\rho = \rho(X, \chi_K)$  is a minimal conductor Heisenberg representation of the absolute Galois group  $G_F$  of a local field  $F/\mathbb{Q}_p$  of dimension  $m$  which is prime to  $p$ . This means we are in the situation:  $\rho = \rho(X, \chi_K) = \rho(X_\eta, \chi_K)$ , where  $\eta$  is a character of  $U_F/U_F^1$ , and  $\dim(\rho) = \#\eta = m$ .

Since  $\rho$  is of minimal conductor, we have  $a(\rho_0) = m$ . Then from Remark 3.24 we have  $a(\chi_K) = 1$ .

Now we choose  $E/F \subset K/F$  a totally ramified cyclic subextension of degree  $[E : F] = m$ , hence  $k_E = k_F$  the same residue fields, and  $K/E$  is unramified of degree  $m$ . Then we can write  $\rho = \text{Ind}_{E/F}(\chi_E)$ , and  $a(\chi_E) = 1$ . Again, from Proposition 3.7 we have

$$\det(\rho)(x) = \Delta_{E/F}(x) \cdot \chi_K \circ N_{K/E}^{-1}(x) \quad \text{for all } x \in F^\times.$$

Then for all  $x \in F^\times$ , we can write

$$\chi_K \circ N_{K/E}^{-1}(x) = \chi_E(x) = \Delta_{E/F}(x) \cdot \det(\rho)(x).$$

This is true for all subextensions<sup>6</sup>  $E/F$  in  $K/F$  which are cyclic of degree  $m$ .

Now we come to in our particular choice:  $\rho = \text{Ind}_{E/F}(\chi_E)$ , with  $a(\chi_E) = 1$  and  $E/F$  is totally ramified. We can write

$$\begin{aligned} W(\rho, \psi) &= W(\text{Ind}_{E/F}(\chi_E), \psi) = \lambda_{E/F}(\psi) \cdot W(\chi_E, \psi \circ \text{Tr}_{E/F}) \\ &= \lambda_{E/F}(\psi) \cdot q_E^{-\frac{1}{2}} \chi_E(c_E) \sum_{x \in U_E/U_E^1} \chi_E^{-1}(x) (c_E^{-1} \psi \circ \text{Tr}_{E/F})(x), \end{aligned}$$

where  $v_E(c_E) = 1 + n(\psi \circ \text{Tr}_{E/F}) = e_{E/F}(1 + n(\psi))$ . This implies that we can choose  $c_F \in F^\times$  such that  $\nu_F(c_F = c_E) = 1 + n(\psi)$ . Let  $E_1/F$  be the unramified subextension in  $K/F$ , then for each  $\epsilon \in U_F$ , we have  $N_{K/E}^{-1}(\epsilon) = N_{E_1/F}^{-1}(\epsilon)$  where  $N_{E_1/F} := N_{K/E}|_{E_1}$ . Since  $E/F$  is totally ramified, we have  $q_E = q_F$ . And when  $x \in F^\times$ , we have  $\text{Tr}_{E/F}(x) = mx$ .

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<sup>6</sup>In  $K/F$  of type  $\mathbb{Z}_m \times \mathbb{Z}_m$  any cyclic subextension  $E/F$  in  $K/F$  of degree  $m$  will correspond to a maximal isotropic subgroup. But we restrict to choosing  $E$  totally ramified or unramified.

Then the above formula rewrites:

$$\begin{aligned}
W(\rho, \psi) &= \lambda_{E/F}(\psi) \cdot q_F^{-\frac{1}{2}} \chi_K \circ N_{K/E}^{-1}(c_F) \sum_{x \in k_F^\times} (\chi_K \circ N_{K/E}^{-1})^{-1}(x) (c_F^{-1} \psi)(mx) \\
&= \lambda_{E/F}(\psi) \cdot q_F^{-\frac{1}{2}} \Delta_{E/F}(c_F) \det(\rho)(c_F) \sum_{x \in k_F^\times} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(x) (c_F^{-1} \psi)(mx) \\
&= \lambda_{E/F}(\psi) \Delta_{E/F}(c_F) \cdot \left( \det(\rho)(c_F) q_F^{-\frac{1}{2}} \sum_{x \in k_F^\times} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(x) (c_F^{-1} \psi)(mx) \right) \\
&= R(\psi, c) \cdot L(\psi, c),
\end{aligned}$$

where  $c_F = c \in F^\times$  with  $\nu_F(c) = 1 + n(\psi)$ ,  $R(\psi, c) = \lambda_{E/F}(\psi) \Delta_{E/F}(c)$ , and

$$L(\psi, c) = \det(\rho)(c_F) q_F^{-\frac{1}{2}} \sum_{x \in k_F^\times} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(x) (c^{-1} \psi)(mx).$$

Now it is clear that  $L(\psi, c)$  depends on  $c$  but not on the totally ramified cyclic extension  $E/F$  which we have chosen.

Again we know that  $\lambda_{E/F}(\psi)$  is a fourth root of unity and  $\Delta_{E/F}(c) \in \{\pm 1\}$ . Therefore it is easy to see that  $R(\psi, c)$  is a fourth root of unity. So to call our expression

$$W(\rho, \psi) = R(\psi, c) \cdot L(\psi, c)$$

is invariant, we are left to show  $R(\psi, c)$  does not depend on the the totally ramified cyclic subextension  $E/F$  in  $K/F$ .

Moreover, we can write (cf. Lemma 3.2 of [25]) here

$$R(\psi, c) = \lambda_{E/F}(\psi) \Delta_{E/F}(c) = \lambda_{E/F}(c\psi) = \lambda_{E/F}(\psi'),$$

where  $\psi' = c\psi$ , hence  $n(\psi') = \nu_F(c) + n(\psi) = 1 + n(\psi) + n(\psi) = 2n(\psi) + 1$ .

When  $m(= [E : F])$  is odd, we have  $\lambda_{E/F}(\psi') = 1$ , hence  $R(\psi, c) = \lambda_{E/F}(c\psi) = 1$ . Thus in the odd case  $R(\psi, c)$  is independent of the choice of the totally ramified subextension  $E/F$  in  $K/F$ .

When  $m$  is even, we have

$$\begin{aligned}
R(\psi, c) &= \lambda_{E/F}(\psi') = \lambda_{E/E'}(\psi'') \cdot \lambda_{E'/F}^{[E:E'']} \\
&= \lambda_{E'/F}(\psi')^{\pm 1},
\end{aligned}$$

where  $[E', F]$  is the 2-primary part of  $m$ , hence  $[E : E']$  is odd. Here the sign only depends on  $m$  but not on  $E$ . So we can restrict to the case where  $m = [E : F]$  is a power of 2. Let  $E_2/F$  be the unique quadratic subextension in  $E/F$ . Since  $E/F$  is a cyclic tame extension, from Theorem 2.5, we obtain:

$$(4.13) \quad \lambda_{E/F}(\psi') = \begin{cases} \lambda_{E_2/F}(\psi') & \text{if } [E : F] \neq 4 \\ \beta(-1) \cdot \lambda_{E_2/F}(\psi') & \text{if } [E : F] = 4, \end{cases}$$

where  $\beta$  is the character of  $F^\times / \mathcal{N}_{E/F}$  of order 4.

Since here  $n(\psi') = 2n(\psi) + 1$  is **odd**<sup>7</sup>, from Remark 5.10 of [25] we can tell that  $\lambda_{E_2/F}(\psi')$  is invariant.

Finally, we have to see that  $\beta(-1)$  does not depend on  $E$  if  $[E : F] = 4$ .

Since  $E/F$  is totally ramified of degree 4, we have  $F^\times = U_F \cdot N$ , hence  $F^\times/N = U_F N/N = U_F/U_F \cap N \cong \mathbb{Z}_4$ , where  $N = N_{E/F}(E^\times)$ . Again  $U_F^1 \subset U_F$ , and  $U_F^1 \subset N$ , hence  $U_F^1 \subset N \cap U_F \subset U_F$ . We know that  $U_F/U_F^1$  is a cyclic group. Therefore  $N \cap U_F$  is determined by its index in  $U_F$ , which does not depend on  $E$ . Hence,  $U_F \cap N$  does not depend on  $E$ .

We also know that there are two characters of  $U_F/U_F \cap N$  of order 4, and they are inverse to each other. Then

$$\beta(-1) = \beta(-1)^{-1} = \beta^{-1}(-1)$$

is the same in both cases. Since  $\beta$  is the character which corresponds to  $E/F$  by class field theory, we can say  $\beta$  is a character of  $F^\times/U_F^1$ , hence  $a(\beta) = 1$ . It clearly shows that  $\beta(-1)$  does not depend on  $E$ . So we can conclude that  $R(\psi, c)$  does not depend on  $E$ .

Thus our expression  $W(\rho, \psi) = R(\psi, c) \cdot L(\psi, c)$  does not depend on the choice of the totally ramified cyclic subextension  $E/F$  in  $K/F$ . Moreover we notice that we have the transformation rules

$$R(\psi, \epsilon c) = \Delta(\epsilon)R(\psi, c), \quad L(\psi, \epsilon c) = \Delta(\epsilon)L(\psi, c),$$

for all  $\epsilon \in U_F$ . Again  $\Delta(\epsilon)^2 = 1$ , hence the product  $R(\psi, \epsilon c) \cdot L(\psi, \epsilon c) = R(\psi, c) \cdot L(\psi, c) = W(\rho, \psi)$  does not depend on the choice of  $c$ .

Therefore, finally, we can conclude our formula  $W(\rho, \psi) = R(\psi, c)L(\psi, c)$  is an invariant expression.

□

Now let  $\rho = \rho(X, \chi_K)$  be a Heisenberg representation of dimension prime to  $p$  but the conductor of  $\rho$  is **not** minimal. In the following theorem we give an invariant formula of  $W(\rho, \psi)$ .

**Theorem 4.9.** *Let  $\rho = \rho(X_\rho, \chi_K)$  be a Heisenberg representation of the absolute Galois group  $G_F$  of a local field  $F/\mathbb{Q}_p$  of dimension  $m$  prime to  $p$ . Let  $\psi$  be a nontrivial additive character of  $F$ . Suppose that the conductor of  $\rho$  is not minimal,  $\rho = \rho_0 \otimes \widetilde{\chi}_F$  and  $a(\rho) = m \cdot a(\chi_F)$ , where  $\widetilde{\chi}_F : W_F \rightarrow \mathbb{C}^\times$  corresponds to  $\chi_F : F^\times \rightarrow \mathbb{C}^\times$ , and  $h = a(\chi_F) \geq 2$ .*

**Case-1:** *If  $m$  is odd, then*

(1) *when  $1 + m(h - 1) = 2d$  is even, we have*

$$W(\rho, \psi) = \det(\rho)(c)\psi(mc^{-1}),$$

(2) *when  $1 + m(h - 1) = 2d + 1$  is odd, we have*

$$W(\rho, \psi) = \det(\rho)(c) \cdot H(\psi, c),$$

where

$$H(\psi, c) = q_F^{-\frac{1}{2}} \sum_{y \in U_F^{h'}/U_F^{h'+1}} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(y)(c^{-1}\psi)(my),$$

<sup>7</sup>If  $n(\psi')$  is even, then from the table of the Remark 5.10 of [25],  $\lambda_{E_2/F}(\psi') = -\lambda_{E'_2/F}(\psi')$ , where  $E'_2/F$  be the totally ramified quadratic extension different from  $E_2/F$ . Therefore  $\lambda_{E/F}(\psi')$  depends on  $\psi'$ .

and  $h' = [\frac{h}{2}]$ , where  $[x]$  denotes the largest integer  $\leq x$ .

**Case-2:** If  $m$  is even, then

(1) when  $h$  is odd, we have

$$W(\rho, \psi) = R(\psi, c) \cdot \det(\rho)(c) \cdot H(\psi, c),$$

where  $H(\psi, c)$  is the same as in Case-1(2).

(2) when  $h$  is even, we have

$$W(\rho, \psi) = R(\psi, c) \cdot \det(\rho)(c) \cdot q_F^{\frac{1}{2}} \cdot \psi(c^{-1}m),$$

where  $R(\psi, c) = \lambda_{E/F}(\psi) \cdot \Delta_{E/F}(c)$ .

Here  $E_1/F$  is the maximal unramified subextension in  $K/F$ , and  $E/F$  is a totally ramified cyclic subextension in  $K/F$  and  $c \in F^\times$  with  $\nu_F(c) = h + n(\psi)$ , and

$$\chi_F(1+x) = \psi(x/c), \quad \text{for all } x \in P_F^{h-h'}/P_F^h.$$

*Proof. Step-1:* By the given condition,  $\dim(\rho) = m$  prime to  $p$ , and the Artin conductor  $a_F(\rho) = mh$  where  $h \geq 2$ , then from Lemma 3.23, we have  $a(\chi_E) = mh - d_{E/F} = mh - m + 1 = 1 + m(h-1)$ , where  $E/F$  is a totally ramified cyclic subextension in  $K/F$ , and  $\rho = \text{Ind}_{E/F}(\chi_E)$ .

Since by the given condition  $\rho$  is not minimal conductor, we can write

$$(4.14) \quad \rho = \rho_0 \otimes \widetilde{\chi}_F,$$

where  $\rho = \rho_0(X, \chi_0)$  is a minimal conductor Heisenberg representation of dimension  $m$ , and  $\widetilde{\chi}_F : W_F \rightarrow \mathbb{C}^\times$  corresponds to  $\chi_F : F^\times \rightarrow \mathbb{C}^\times$  by class field theory.

Then we have  $X_\rho = X_\eta$  for  $\eta : U_F/U_F^1 \rightarrow \mathbb{C}^\times$ ,  $\#\eta = m$  and:

$$\rho_0 = \text{Ind}_{E/F}(\chi_{E,0}) \quad \rho = \text{Ind}_{E/F}(\chi_E),$$

where  $E/F$  is a cyclic totally ramified extension of degree  $m$ .

Because of (4.14) we may assume now that

$$(4.15) \quad \chi_E = \chi_{E,0} \cdot (\chi_F \circ N_{E/F}), \quad a(\chi_{E,0}) = 1, \quad a(\chi_E) = a(\chi_F \circ N_{E/F}) = 1 + m(h-1).$$

From the first and second of the equalities (4.15) we deduce

$$(4.16) \quad \chi_E|_{U_E^1} = (\chi_F \circ N_{E/F})|_{U_E^1}, \quad N_{E/F}(U_E^1) = U_F^1,$$

where the second equality holds because  $E/F$  is totally ramified, and it implies that conversely  $\chi_E|_{U_E^1}$  determines  $\chi_F|_{U_F^1}$ .

**Step-2:** Now for  $d \geq 1$  we put:

$$A_E := U_E^d / U_E^{d+1},$$

which we consider as a  $\text{Gal}(E/F)$ -module. We also know that  $A_E / I_{E/F} A_E \cong A_E^{\text{Gal}(E/F)}$ , where  $I_{E/F} A_E$  is the augmentation with respect to the extension  $E/F$ .

We also know that for any finite extension  $E/F$ , we have

$$(4.17) \quad U_E^d \cap F^\times = \begin{cases} U_F^{\frac{d}{e_{E/F}}} & \text{if } e_{E/F} \text{ divides } d \\ U_F^{[\frac{d}{e_{E/F}}]+1} & \text{if } e_{E/F} \text{ does not divide } d. \end{cases}$$

Again we also have

$$A_E^{\text{Gal}(\mathbb{E}/\mathbb{F})} = U_E^n / U_E^{n+1} \cap F^\times = U_E^d \cap F^\times / U_E^{d+1} \cap F^\times.$$

**Step-3:** If  $1 + m(h-1) = 2d+1$ , then  $\frac{d}{m} = \frac{h-1}{2}$ . Let  $h' := [\frac{h}{2}]$ . If  $A_E = U_E^d / U_E^{d+1}$ , and  $h$  is odd, then we have:

$$U_E^d \cap F^\times / U_E^{d+1} \cap F^\times = U_F^{\frac{h-1}{2}} / U_F^{\frac{h-1}{2}+1} = U_F^{h'} / U_F^{h'+1},$$

and if  $h$  is even, hence 2 does not divide  $h-1$ , then we can write

$$U_E^d \cap F^\times / U_E^{d+1} \cap F^\times = U_F^{[\frac{h-1}{2}]+1} / U_F^{[\frac{h-1}{2}]+1} \cong \{1\}.$$

Since  $A_E^{\text{Gal}(\mathbb{E}/\mathbb{F})} \cong A_E / I_{E/F} A_E$ , we can uniquely write any element  $x \in U_E^d / U_E^{d+1}$  as  $x = yz$  where  $y \in A_E^{\text{Gal}(\mathbb{E}/\mathbb{F})}$  and  $z \in I_{E/F} A_E$ . We also know that  $U_E^d / U_E^{d+1} \cong k_E$ , hence  $|A_E| = |A_E^{\text{Gal}(\mathbb{E}/\mathbb{F})}| \cdot |I_{E/F} A_E| = q_E = q_F$ . We also observe that when  $h$  is even, we have  $A_E^{\text{Gal}(\mathbb{E}/\mathbb{F})} \cong \{1\}$ , hence  $|A_E| = |I_{E/F} A_E| = q_F$ . And when  $h$  is odd, we have  $A_E^{\text{Gal}(\mathbb{E}/\mathbb{F})} = U_F^{h'} / U_F^{h'+1}$ , and hence  $|A_E^{\text{Gal}(\mathbb{E}/\mathbb{F})}| = q_F$ . So this implies  $|I_{E/F} A_E| = 1$ .

Now set:

$$S(\psi, c) := \sum_{x \in A_E} \chi_E^{-1}(x) (c^{-1} \psi) (\text{Tr}_{\mathbb{E}/\mathbb{F}}(x)).$$

Then we can write

$$\begin{aligned} S(\psi, c) &= \sum_{y \in A_E^{\text{Gal}(\mathbb{E}/\mathbb{F})}, z \in I_{E/F} A_E} \chi_E^{-1}(yz) \cdot (c^{-1} \psi) (\text{Tr}_{\mathbb{E}/\mathbb{F}}(yz)) \\ &= \sum_{y \in A_E^{\text{Gal}(\mathbb{E}/\mathbb{F})}} \sum_{z \in I_{E/F} A_E} \chi_E^{-1}(yz) (c^{-1} \psi) (\text{Tr}_{\mathbb{E}/\mathbb{F}}(yz)) \\ &= |I_{E/F} A_E| \cdot \sum_{y \in A_E^{\text{Gal}(\mathbb{E}/\mathbb{F})}} \chi_E^{-1}(y) (c^{-1} \psi) (my) \\ &= |I_{E/F} A_E| \cdot \sum_{y \in A_E^{\text{Gal}(\mathbb{E}/\mathbb{F})}} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(y) (c^{-1} \psi) (my) \\ &= \begin{cases} \sum_{y \in U_F^{h'} / U_F^{h'+1}} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(y) (c^{-1} \psi) (my) & \text{when } h \text{ is odd} \\ q_F \cdot (c^{-1} \psi)(m) = q_F \cdot \psi(mc^{-1}) & \text{when } h \text{ is even,} \end{cases} \end{aligned}$$

since  $\chi_E(yz) = \chi_E(y)$  and  $\text{Tr}_{\mathbb{E}/\mathbb{F}}(yz) = y \text{Tr}_{\mathbb{E}/\mathbb{F}}(z) = ym$ .

**Step-4:** Again, we have  $\rho = \text{Ind}_{\mathbb{E}/\mathbb{F}}(\chi_E)$ . Then

$$W(\rho, \psi) = W(\text{Ind}_{\mathbb{E}/\mathbb{F}}(\chi_E), \psi) = \lambda_{\mathbb{E}/\mathbb{F}}(\psi) \cdot W(\chi_E, \psi \circ \text{Tr}_{\mathbb{E}/\mathbb{F}}).$$

**Case-1: Suppose that  $m$  is odd:**

**(1) When  $a(\chi_E) = 1 + m(h-1) = 2d$ :** In this situation,  $h$  must be even and we take  $h = 2h'$ , hence  $d = mh' - \frac{m-1}{2}$ . Since  $m(h'-1) < d \leq mh'$ , we have  $P_E^d \cap F = P_F^{h'}$ . Now we choose  $c \in F^\times$  such that

$$(4.18) \quad \chi_F(1+y) = \psi(c^{-1}y), \quad \text{for all } y \in P_F^{h-h'} / P_F^h,$$

hence  $\nu_F(c) = a(\chi_F) + n(\psi) = h + n(\psi)$ . Now if we take an element  $y_E \in P_E^{a(\chi_E)-d} = P_E^d$ , then  $\text{Tr}_{E/F}(y_E) \in \mathbf{P}_F^{h'} = \mathbf{P}_F^{h-h'}$  because  $m(h' - 1) < d \leq mh' = m(h - h')$ . Since  $E/F$  is cyclic, from Proposition 1.1 on p. 68 of [12], we have:

$$N_{E/F}(1 + y_E) = 1 + \text{Tr}_{E/F}(y_E) + N_{E/F}(y_E) + \text{Tr}_{E/F}(\delta),$$

where  $\nu_E(\delta) \geq 2d = a(\chi_E)$ . Then for all  $y_E \in P_E^{a(\chi_E)-d}/P_E^{a(\chi_E)}$ , we can write

$$\begin{aligned} \chi_E(1 + y_E) &= \chi_F \circ N_{E/F}(1 + y_E) = \chi_F(1 + \text{Tr}_{E/F}(y_E)) \\ (4.19) \quad &= \psi(c^{-1} \text{Tr}_{E/F}(y_E)) = (c^{-1} \psi_E)(y_E), \end{aligned}$$

because  $N_{E/F}(y_E) + \text{Tr}_{E/F}(\delta) \in \mathbf{P}_F^h$ . This verifies that our choice of  $c$  is right for applying Lamprecht-Tate formula for  $W(\chi_E, \psi_E)$ .

Now we apply Lamprecht-Tate formula (cf. Theorem 6.1.1 and its Corollary of [27]) and we obtain:

$$W(\chi_E, \psi_E) = \chi_E(c) \cdot (c^{-1} \psi_E)(1) = \Delta_{E/F}(c) \det(\rho)(c) \psi(mc^{-1}).$$

Therefore

$$\begin{aligned} W(\rho, \psi) &= \lambda_{E/F}(\psi) \cdot W(\chi_E, \psi_E) \\ &= \lambda_{E/F}(\psi) \cdot \Delta_{E/F}(c) \det(\rho)(c) \psi(mc^{-1}) \\ &= R(\psi, c) \cdot \det(\rho)(c) \cdot \psi(mc^{-1}) \\ &= \det(\rho)(c) \cdot \psi(mc^{-1}), \end{aligned}$$

where  $R(\psi, c) = \lambda_{E/F}(\psi) \Delta_{E/F}(c) = \lambda_{E/F}(c\psi) = 1$  because  $E/F$  is an odd degree Galois extension.

**(2).** When  $a(\chi_E) = 1 + m(h - 1) = 2d + 1$ : Since  $m$  is odd, here  $h$  must be odd. Let  $h' := \lfloor \frac{h}{2} \rfloor$ . Then from Step-3 we have  $A_E^{\text{Gal}(E/F)} = U_F^{h'}/U_F^{h'+1}$ . Now if we choose  $c \in F^\times$  such that

$$\chi_F(1 + y) = \psi(c^{-1}y), \quad \text{for all } y \in P_F^{h-h'}/P_F^h.$$

Then this  $c$  also satisfies the following relation

$$\chi_E(1 + y_E) = \psi_E(c^{-1}y_E), \quad \text{for all } y_E \in P_E^{a(\chi_E)-d}/P_E^{a(\chi_E)},$$

because  $d = \frac{m(h-1)}{2}$ , and hence  $m(h' - 1) < d \leq mh'$ . Then by Lamprecht-Tate formula we have

$$\begin{aligned} W(\chi_E, \psi_E) &= \chi_E(c) \psi_E(c^{-1}) q_E^{-\frac{1}{2}} \sum_{x \in P_E^d/P_E^{d+1}} \chi_E^{-1}(1 + x) \cdot (c^{-1} \psi_E)(x) \\ &= \chi_E(c) \cdot q_F^{-\frac{1}{2}} \sum_{x \in U_E^d/U_E^{d+1}} \chi_E^{-1}(x) \cdot (c^{-1} \psi)(\text{Tr}_{E/F}(x)) \\ &= \Delta_{E/F}(c) \det(\rho)(c) q_F^{-\frac{1}{2}} \sum_{y \in U_F^{h'}/U_F^{h'+1}} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(y) (c^{-1} \psi)(my), \end{aligned}$$

because  $h$  is odd, and we use Step-3. Thus we obtain

$$\begin{aligned}
W(\rho, \psi) &= W(\text{Ind}_{E/F}(\chi_E), \psi) = \lambda_{E/F}(\psi) \cdot W(\chi_E, \psi_E) \\
&= \lambda_{E/F}(\psi) \cdot \Delta_{E/F}(c) \det(\rho)(c) q_F^{-\frac{1}{2}} \sum_{y \in U_F^{h'}/U_F^{h'+1}} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(y) (c^{-1}\psi)(my). \\
&= R(\psi, c) \cdot \det(\rho)(c) q_F^{-\frac{1}{2}} \sum_{y \in U_F^{h'}/U_F^{h'+1}} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(y) (c^{-1}\psi)(my). \\
&= \det(\rho)(c) q_F^{-\frac{1}{2}} \sum_{y \in U_F^{h'}/U_F^{h'+1}} (\chi_K \circ N_{E_1/F}^{-1})^{-1}(y) (c^{-1}\psi)(my),
\end{aligned}$$

because  $m$  is odd, hence  $R(\psi, c) = \lambda_{E/F}(c\psi) = 1$ .

**Case-2: Suppose that  $m$  is even.** If  $m$  is even, then  $1 + m(h-1) = 2d+1$  is always an odd number and  $d = \frac{m(h-1)}{2}$ . But here  $h$  could be any number  $\geq 2$ , i.e.,  $h$  is not fixed, and we put  $h' := [\frac{h}{2}]$ . This implies  $m(h'-1) < d \leq mh'$  and  $P_E^d \cap F = P_F^{h'}$ . Now we take  $c \in F^\times$  such that (4.18) holds, and this again satisfies equation (4.19). Therefore we can use Lamprecht-Tate formula and we have two cases:

(1) When  $h$  is odd, we are in the same situation of Case-1(2), and we have

$$W(\rho, \psi) = R(\psi, c) \cdot \det(\rho)(c) \cdot H(\psi, c).$$

(2) When  $h$  is even, from Step-3 we know that  $A_E^{\text{Gal}(E/F)} \cong \{1\}$  and

$$\sum_{x \in A_E} \chi_E^{-1}(x) (c^{-1}\psi)(\text{Tr}_{E/F}(x)) = q_F \cdot \psi(mc^{-1}).$$

Therefore in this situation we have

$$\begin{aligned}
W(\rho, \psi) &= R(\psi, c) \cdot \det(\rho)(c) q_F^{-\frac{1}{2}} \cdot \sum_{x \in A_E} \chi_E^{-1}(x) (c^{-1}\psi)(\text{Tr}_{E/F}(x)) \\
&= R(\psi, c) \cdot \det(\rho)(c) q_F^{-\frac{1}{2}} \cdot q_F \psi(mc^{-1}) \\
&= R(\psi, c) \cdot \det(\rho)(c) q_F^{\frac{1}{2}} \psi(mc^{-1}).
\end{aligned}$$

Furthermore, in the proof of Theorem 4.7, we observe that  $R(\psi, c)$  does not depend on  $E$ . Hence our above computations are invariant.

This completes the proof. □

By using following lemma, without using  $\lambda$ -function we also can give invariant formula for  $W(\rho)$ , when  $\dim(\rho)$  is prime to  $p$ , for sufficiently large conductor character  $\chi_F$ .

**Lemma 4.10 (Deligne-Henniart, [2], p. 190, Proposition 29.4(4)).** *Let  $F$  be a non-archimedean local field and  $\psi$  be a nontrivial additive character of  $F$ . Let  $\rho$  be a finite dimensional representation of  $G_F$ . There is a sufficiently large integer  $m_\rho$  such that if  $\chi_F$  is a character of  $F^\times$  of conductor  $a(\chi_F) \geq m_\rho$ , then*

$$(4.20) \quad W(\rho \otimes \chi_F, \psi) = W(\chi_F, \psi)^{\dim(\rho)} \cdot \det(\rho)(c),$$



for any  $c := c(\chi_F, \psi) \in F^\times$  such that  $\chi_F(1+x) = \psi(c^{-1}x)$ ,  $x \in P_F^{\lfloor \frac{a(\chi_F)}{2} \rfloor + 1}$ .

By using the above Lemma 4.10, we obtain the following theorem.

**Theorem 4.11.** *Let  $\rho = \rho_0 \otimes \widetilde{\chi}_F$  be a Heisenberg representation of  $G_F$  of dimension  $d$  with  $\gcd(d, p) = 1$ , where  $\rho_0 = \rho_0(X_\eta, \chi_0)$  is a minimal conductor Heisenberg representation. If  $a(\chi_F) \geq m_\rho \geq 2$ , a sufficiently large number which depends on  $\rho$ , then we have*

$$(4.21) \quad W(\rho, \psi) = W(\rho_0 \otimes \widetilde{\chi}_F) = W(\chi_F, \psi)^d \cdot \det(\rho_0)(c),$$

where  $\psi$  is a nontrivial additive character of  $F$ , and  $c := c(\chi_F, \psi) \in F^\times$ , satisfies

$$\chi_F(1+x) = \psi(c^{-1}x) \text{ for all } x \in P_F^{\lfloor \frac{a(\chi_F)}{2} \rfloor + 1}.$$

*Proof.* From Corollary 3.12 we know that all Heisenberg representation  $\rho$  of  $G_F$  of dimension prime to  $p$  are precisely given as  $\rho = \rho(X_\eta, \chi)$  for characters  $\eta$  of  $U_F/U_F^1$ . Then from Remark 3.24 we have here  $a_K(\chi_0) = 1$ . This implies that we always can choose a character  $\chi_0$  of  $K^\times$  with  $a(\chi_0) = 1$  such that all other  $\chi_K$  are given as

$$\chi_K = (\chi_F \circ N_{K/F}) \cdot \chi_0,$$

for arbitrary characters  $\chi_F$  of  $F^\times$ . Therefore the whole set of Heisenberg (U-isotopic) representations of  $G_F$  of dimension prime to  $p$  is:

$$\rho_0 = \rho_0(G_K, \chi_0) \text{ and } \rho = \rho(G_K, \chi_K), \text{ where } \chi_K = (\chi_F \circ N_{K/F}) \cdot \chi_0, \text{ and } \chi_F \in \widehat{F^\times}.$$

We also know that there are  $d^2$  characters of  $F^\times/F^{\times d}$  such that  $\rho_0 \otimes \widetilde{\chi} = \rho_0$  (cf. [5], p. 303, Proposition 1.4). So we always have:

$$\rho = \rho_0 \otimes \widetilde{\chi}_F = \rho_0 \otimes \widetilde{\chi\chi_F},$$

where  $\chi \in \widehat{F^\times/F^{\times d}}$ , and  $\widetilde{\chi}_F : W_F \rightarrow \mathbb{C}^\times$  corresponds to  $\chi_F$  by class field theory.

Let  $\zeta$  be a  $(q_F - 1)$ -st root of unity. Since  $U_F^1$  is a pro- $p$ -group and  $\gcd(p, d) = 1$ , we have

$$(4.22) \quad F^\times/F^{\times d} = \langle \pi_F \rangle \times \langle \zeta \rangle \times U_F^1 / \langle \pi_F^d \rangle \times \langle \zeta \rangle^d \times U_F^1 \cong \mathbb{Z}_d \times \mathbb{Z}_d,$$

that is, a direct product of two cyclic group of same order. Hence  $F^\times/F^{\times d} \cong \widehat{F^\times/F^{\times d}}$ . Since  $F^{\times d} = \langle \pi_F^d \rangle \times \langle \zeta \rangle^d \times U_F^1$ , and  $F^\times/F^{\times d} \cong \mathbb{Z}_d \times \mathbb{Z}_d$ , we have  $a(\chi) \leq 1$  and  $\#\chi$  is a divisor of  $d$  for all  $\chi \in \widehat{F^\times/F^{\times d}}$ . Now if we take a character  $\chi_F$  of  $F^\times$  conductor  $\geq m_\rho \geq 2$ , hence  $a(\chi_F) \geq 2a(\chi)$  for all  $\chi \in \widehat{F^\times/F^{\times d}}$ . Then by using Deligne's formula (cf. [28], Lemma 4.16) we have

$$W(\chi_F \chi, \psi)^d = \chi(c)^d \cdot W(\chi_F, \psi)^d = W(\chi_F, \psi)^d,$$

where  $c \in F^\times$  with  $\nu_F(c) = a(\chi_F) + n(\psi)$ , satisfies

$$\chi_F(1+x) = \psi(c^{-1}x), \quad \text{for all } x \in F^\times \text{ with } 2\nu_F(x) \geq a(\chi).$$

Finally, by using Lemma 4.10 we can write

$$\begin{aligned} W(\rho, \psi) &= W(\rho_0 \otimes \widetilde{\chi_F \chi}, \psi) = W(\chi_F \chi, \psi)^{\dim(\rho_0)} \cdot \det(\rho_0)(c(\chi_F, \psi)) \\ &= W(\chi_F, \psi)^d \cdot \det(\rho_0)(c). \end{aligned}$$

□

### 5. Applications of Tate's root-of-unity criterion

Let  $K/F$  be a finite Galois extension of the non-archimedean local field  $F$ , and  $\rho : \text{Gal}(K/F) \rightarrow \text{Aut}_{\mathbb{C}}(V)$  a representation of  $\text{Gal}(K/F)$  on a complex vector space  $V$ . Let  $P(K/F)$  denote the first **wild** ramification group of  $K/F$ . Let  $V^P$  be the subspace of all elements of  $V$  fixed by  $\rho(P(K/F))$ . Then  $\rho$  induces a representation:

$$\rho^P : \text{Gal}(K/F)/P(K/F) \rightarrow \text{Aut}_{\mathbb{C}}(V^P).$$

Let  $\bar{F}$  be an algebraic closure of the local field  $F$ , and  $G_F = \text{Gal}(\bar{F}/F)$  be the absolute Galois group for  $\bar{F}/F$ . Let  $\rho$  be a representation of  $G_F$ .

**Then by Tate,  $W(\rho)/W(\rho^P)$  is a root of a unity (cf. [19], p. 112, Corollary 4).**

Now let  $\rho$  be an irreducible representation  $G_F$ , then either  $\rho^P = \rho$ , in which case  $\frac{W(\rho)}{W(\rho^P)} = 1$ , or else  $\rho^P = 0$ , in this case from Tate's result we can say  $W(\rho)$  is a root of unity. Equivalently: If  $W(\rho)$  is not a root of unity then  $\rho^P \neq 0$ , hence  $\rho^P = \rho$  because  $\rho$  is irreducible. This means that all vectors  $v \in V$  of the representation space are fixed under  $P$  action on  $V$ .

In other words, if we consider  $\rho$  as a homomorphism  $\rho : G_F \rightarrow \text{Aut}_{\mathbb{C}}(V)$  then the elements from  $P$  are mapped to the identity, hence

$$\rho^P = \rho \text{ means } P \subset \text{Ker}(\rho).$$

Therefore we can state the following lemma.

**Lemma 5.1.** *If  $\rho$  is an irreducible representation of  $G_F$ , such that the subgroup  $P \subset G_F$ , of wild ramification does **not trivially** act on the representation space  $V$  (this gives  $\rho^P \neq \rho$ , i.e.,  $\rho^P = 0$ ), then  $W(\rho)$  is a root of unity.*

Before going to our next results we need to recall some facts from class field theory. Let  $F$  be a non-archimedean local field. Let  $F^{ab}$  be the maximal abelian extension of  $F$  and  $F_{nr}$  be the maximal unramified extension of  $F$ . Then by local class field theory there is a unique homomorphism

$$\theta_F : F^\times \rightarrow \text{Gal}(F^{ab}/F)$$

having certain properties (cf. [18], p. 20, Theorem 1.1). This local reciprocity map  $\theta_F$  is continuous and injective with dense image. From class field theory we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & U_F & \rightarrow & F^\times & \xrightarrow{v_F} & \mathbb{Z} \rightarrow 0 \\ & & \downarrow \theta_F & & \downarrow \theta_F & & \downarrow \text{id} \\ 0 & \rightarrow & I_F & \rightarrow & \text{Gal}(F^{ab}/F) & \rightarrow & \widehat{\mathbb{Z}} \rightarrow 0, \end{array}$$

where  $I_F := \text{Gal}(F^{ab}/F_{nr})$  is the inertia subgroup of  $\text{Gal}(F^{ab}/F)$ , and  $\text{Gal}(F_{nr}/F)$  is identified with  $\widehat{\mathbb{Z}}$  (cf. [17], p. 144). We also know that  $\theta_F : U_F \rightarrow I_F$  is an isomorphism. Moreover the descending chain

$$U_F \supset U_F^1 \supset U_F^2 \cdots$$

is mapped isomorphically by  $\theta_F$  to the descending chain of ramification subgroups of  $\text{Gal}(F^{ab}/F)$  in the upper numbering.

Now let  $I$  be the inertia subgroup of  $G_F$ . Let  $P$  be the wild ramification subgroup of  $G_F$ . Then we have  $G_F \supset I \supset P$ . Parallel with this we have  $F^\times \supset U_F \supset U_F^1$ . Then we have

$$(5.1) \quad 1 \rightarrow I/P \cdot [G_F, G_F] \rightarrow G_F/P \cdot [G_F, G_F] \rightarrow G_F/I \rightarrow 1,$$

and parallel

$$(5.2) \quad 1 \rightarrow U_F/U_F^1 \rightarrow F^\times/U_F^1 \rightarrow F^\times/U_F \rightarrow 1.$$

Now by class field theory the left terms of sequences (5.1) and (5.2) are isomorphic, but for the right terms we have  $G_F/I$  is isomorphic to the total completion of  $\mathbb{Z}$  (because here  $G_F/I$  is profinite group, hence compact). We also have  $F^\times/U_F = \langle \pi_F \rangle \times U_F/U_F \cong \mathbb{Z}$ . Therefore sequence (5.2) is dense in (5.1) because  $\mathbb{Z}$  is dense in the total completion  $\widehat{\mathbb{Z}}$ . But  $\mathbb{Z}$  and  $\widehat{\mathbb{Z}}$  have the same finite factor groups. **As a consequence  $F^\times/U_F^1$  is also dense in  $G_F/P \cdot [G_F, G_F]$ .**

Let  $\rho$  be a Heisenberg representation of the absolute Galois group  $G_F$ . In the following proposition we show that if  $W(\rho)$  is not a root of unity, then  $\dim(\rho)|(q_F - 1)$ , and  $a_F(\rho)$  is not minimal.

**Proposition 5.2.** *Let  $F/\mathbb{Q}_p$  be a local field and let  $q_F = p^s$  be the order of its finite residue field. If  $\rho = (Z_\rho, \chi_\rho) = \rho(X_\rho, \chi_K)$  is a Heisenberg representation of the absolute Galois group  $G_F$  such that  $W(\rho)$  is not a root of unity, then  $\dim(\rho)|(q_F - 1)$  and  $a_F(\rho)$  is not minimal.*

*Proof.* Let  $P$  denote the wild ramification subgroup of  $G_F$ . By Tate's root-of-unity criterion, we know that  $\gamma := \frac{W(\rho)}{W(\rho^P)}$  is a root of unity. If  $W(\rho)$  is not a root of unity, then  $\rho = \rho^P$ , otherwise  $W(\rho)$  must be a root of unity. Again  $\rho^P = \rho$  implies  $P \subset \text{Ker}(\rho) \subset Z_\rho \subset G_F$ . So  $G_F/Z_\rho$  is a quotient of  $G_F/P$ , hence  $F^\times/U_F^1$ .

Moreover, from the dimension formula (3.6), we have

$$\dim(\rho) = \sqrt{[G_F : Z_\rho]} = \sqrt{[K : F]} = \sqrt{[F^\times : \mathcal{N}_{K/F}]},$$

where  $Z_\rho = G_K$  and  $\text{Rad}(X) = \mathcal{N}_{K/F}$ , hence  $F^\times/N$  is a quotient group of  $F^\times/U_F^1$ . Therefore the alternating character  $X_\rho$  induces an alternating character  $X$  on  $F^\times/U_F^1$ . We also know that  $F^\times = \langle \pi_F \rangle \times \langle \zeta \rangle \times U_F^1$ , where  $\zeta$  is a root of unity of order  $q_F - 1$ . This implies  $F^\times/U_F^1 = \langle \pi_F \rangle \times \langle \zeta \rangle$ . So each element  $x \in F^\times/U_F^1$  can be written as  $x = \pi_F^a \cdot \zeta^b$ , where  $a, b \in \mathbb{Z}$ . We now take  $x_1 = \pi_F^{a_1} \zeta^{b_1}, x_2 = \pi_F^{a_2} \zeta^{b_2} \in F^\times/U_F^1$ , where  $a_i, b_i \in \mathbb{Z} (i = 1, 2)$ , then

$$\begin{aligned} X(x_1, x_2) &= X(\pi_F^{a_1} \zeta^{b_1}, \pi_F^{a_2} \zeta^{b_2}) \\ &= X(\pi_F^{a_1}, \zeta^{b_2}) \cdot X(\zeta^{b_1}, \pi_F^{a_2}) \\ &= \chi_\rho([\pi_F^{a_1}, \zeta^{b_2}]) \cdot \chi_\rho([\zeta^{b_1}, \pi_F^{a_2}]). \end{aligned}$$

But this implies  $X^{q_F-1} \equiv 1$  because  $\zeta^{q_F-1} = 1$ , which means that  $X$  is actually an alternating character on  $F^\times/(F^\times^{(q_F-1)} U_F^1)$ , and therefore  $G_F/G_K$  is actually a quotient of  $F^\times/(F^\times^{(q_F-1)} U_F^1)$ . We also know that  $U_F^1$  is a pro-p-group and therefore

$$U_F^1 = (U_F^1)^{q_F-1} \subset F^\times.$$

Thus the cardinality of  $F^\times/(F^\times^{(q_F-1)} U_F^1)$  is  $(q_F - 1)^2$  because

$$F^\times/(F^\times^{(q_F-1)} U_F^1) \cong \mathbb{Z}/(q_F - 1)\mathbb{Z} \times \langle \zeta \rangle \cong \mathbb{Z}_{q_F-1} \times \mathbb{Z}_{q_F-1}.$$

Therefore  $\dim(\rho)$  divides  $q_F - 1$ .

Since  $\dim(\rho) | q_F - 1$ , from Lemma 3.27 the alternating character  $X_\rho$  is U-isotropic and  $X_\rho = X_\eta$  for a character  $\eta : U_F/U_F^1 \rightarrow \mathbb{C}^\times$ . Since  $\rho = \rho(X_\eta, \chi_K)$  is U-isotropic, from Proposition 3.21,  $a_F(\rho)$  is a multiple of  $\dim(\rho)$ . Moreover, by the given condition,  $W(\rho)$  is not a root of unity, hence  $a_F(\rho)$  is not minimal, otherwise if  $a_F(\rho)$  is minimal, then from Lemma 4.5  $W(\rho)$  is a root of unity.

□

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